PC11 Trigonometry Lesson Solutions

Last year you learned about right-angle trigonometry: SOH CAH TOA. This year in Pre-Calculus 11 you will learn how to solve non right-angle triangles using the Sine Law and the Cosine Law. Do your best to understand this year's trigonometry concepts because you will learn more about trigonometry next year.

- Use of sine and cosine laws to solve non-right triangles, including ambiguous cases
- Contextual and non-contextual problems
- Angles in standard position
- Degrees
- Special angles, as connected with the 30-60-90 and 45-45-90 triangles
- Unit circle
- Reference and co-terminal angles
- Terminal arm
- Trigonometric ratios
- Simple trigonometric equations

1. Label the location of the four quadrants

II	*I*
III	*IV*

2. In which Quadrant is $\boldsymbol{\theta}$ located?
   a. $\boldsymbol{\theta = 120^\circ}$
      II

   b. $\boldsymbol{\theta = -45^\circ}$
      IV

   c. $\boldsymbol{\theta = 400^\circ}$
      I

   d. $\boldsymbol{\theta = -1100^\circ}$
      IV

3. $\boldsymbol{\theta = 300^\circ}$
   a. What is the reference angle?
      $\boldsymbol{60^\circ}$

   b. Find a positive coterminal angle to $\boldsymbol{\theta = 300^\circ}$
      ex. $\boldsymbol{660^\circ}$

   c. Find a negative coterminal angle to $\boldsymbol{\theta = 300^\circ}$
      $\boldsymbol{-60^\circ}$

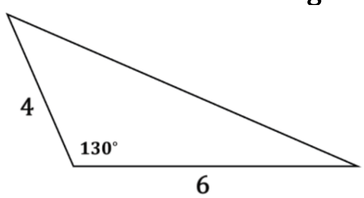
4. Enrichment: Radians vs. Degrees
   This year we measure the angle $\boldsymbol{\theta}$ in degrees. Next year we use a different unit called radians.
   One full revolution $= 2\pi$ radians $= 360°$
   a. Convert π radians to degrees
      $\mathbf{180°}$

   b. Convert $\frac{\pi}{4}$ radians to degrees
      $\mathbf{45°}$

   c. Convert $\frac{\pi}{6}$ radians to degrees
      $\mathbf{30°}$

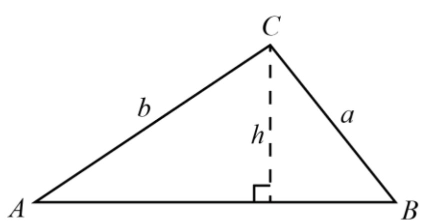
5. Enrichment:
   a. Show that the area of a triangle is $A_\Delta = \frac{1}{2}ab\sin C$
      $\sin C = \frac{h}{a} \rightarrow h = a\sin C$
      $A_\Delta = \frac{bh}{2} = \frac{ba\sin C}{2} = \frac{1}{2}ab\sin C$

   b. Find the area of the triangle below:



      $A = \frac{1}{2}(4)(6)\sin 130° \approx 9.19$

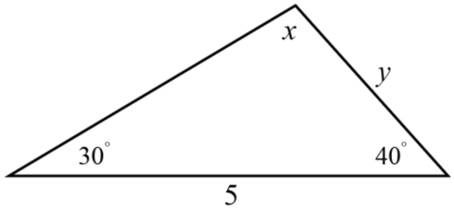
6. Enrichment:
   a. Use the triangle below to help you prove the Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



      Focus on the right triangle: $\sin B = \frac{h}{a} \rightarrow h = a\sin B$
      Now focus on the left triangle: $\sin A = \frac{h}{b} \rightarrow h = b\sin A$
      Thus, $a\sin B = b\sin A$
      Divide both sides by a
      $\sin B = \frac{b\sin A}{a}$
      Divide both sides by b
      $\frac{\sin B}{b} = \frac{\sin A}{a}$

   b. Given the previous proof, why does it follow that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$?
      Take the reciprocals of both sides

7. Solve x and y the ASA triangle below:

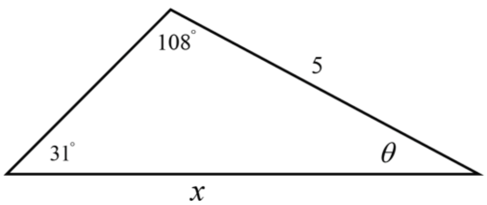


$\boldsymbol{x = 110°}$

$\boldsymbol{\frac{y}{\sin 30°} = \frac{5}{\sin 110°}}$

$\boldsymbol{y \approx 2.66}$ Type equation here.

8. Solve x and θ in the following AAS triangle below:

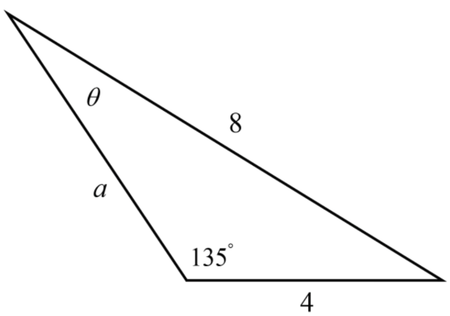


$\boldsymbol{\theta = 41°}$

$\boldsymbol{\frac{x}{\sin 108°} = \frac{5}{\sin 31°}}$

$\boldsymbol{x \approx 9.23}$

9. Solve θ and a in the following SSA triangle below:



$\boldsymbol{\frac{\sin \theta}{4} = \frac{\sin 135°}{8}}$

$\boldsymbol{sin\ \theta \approx 0.35355\ldots}$

$\boldsymbol{\theta \approx 20.7°}$

Unknown angle is $\boldsymbol{\approx 24.3°}$

$\boldsymbol{\frac{a}{\sin 24.3°} = \frac{8}{\sin 135°}}$

$\boldsymbol{a \approx 4.65}$

10. No diagram: Solve the following triangle:

$\boldsymbol{\angle C = 140°, b = 6, c = 30}$

$\boldsymbol{\frac{\sin B}{6} = \frac{\sin 140°}{30}}$

$\boldsymbol{\sin B \approx 0.1285575219}$

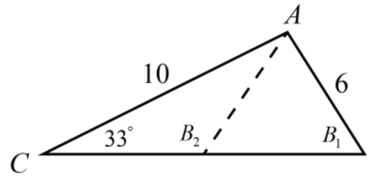
$\boldsymbol{B_1 \approx 7.4°}$

$\boldsymbol{B_2 \approx 172.6°}$ (reject)

$\boldsymbol{a \approx 25.15}$

$\boldsymbol{\angle A \approx 32.6°}$

11. Consider the ambiguous case:
    $\angle C = 33^\circ$. Side $c = 6$. Side $b = 10$.
    a. What are the possible angles of B?



$$\frac{\sin B}{10} = \frac{\sin 33^\circ}{6}$$
$B_1 \approx 65^\circ$
$B_2 \approx 115^\circ$

    b. What are the possible lengths of a?
    $a_1 \approx 10.90$
    $a_2 \approx 5.87$

12. Enrichment: State the number of possible triangles that can be formed. Confirm your answer with an online triangle calculator.
    a. $\angle B = 32^\circ, a = 27, b = 22$
    2

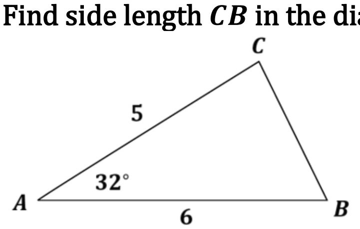
    b. $\angle B = 96^\circ, b = 25, a = 6$
    1

    c. $\angle B = 34^\circ, a = 23, b = 7$
    0

    d. $\angle A = 30^\circ, AC = 8, BC = 5$
    2

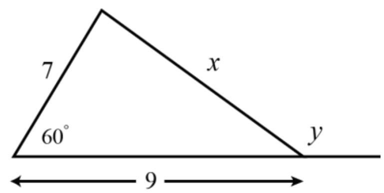
13. When solving a non-right-angled triangle, when should the Sine Law vs. Cosine Law be used?
    Use Sine Law unless you have SSS or SAS (then use Cosine Law)

14. Find side length CB in the diagram below:



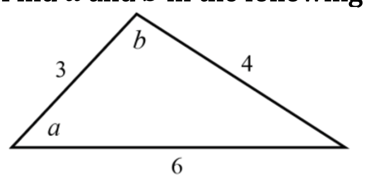
$x^2 = 5^2 + 6^2 - 2(5)(6)\cos 32^\circ$
≈ 3.18

15. Find x and y in the following SAS triangle:



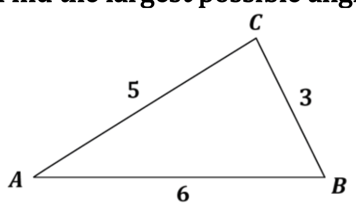
$x^2 = 7^2 + 9^2 - 2(7)(9)\cos 60°$
$x \approx 8.19$
$y \approx 132.2°$

16. Find a and b in the following SSS triangle:



$a \approx 36.3°$ and $b \approx 117.3°$

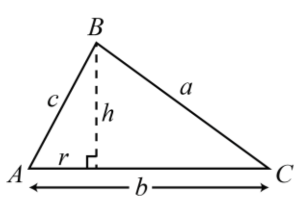
17. Find the largest possible angle in the diagram below:



The largest possible angle is opposite the largest side
$6^2 = 5^2 + 3^2 - 2(5)(3)\cos C$
$\angle C \approx 93.8°$

18. Given $c^2 = a^2 + b^2 - 2ab\cos C$, find an expression for $\angle C$
$2ab\cos C = a^2 + b^2 - c^2$
$\cos C = \frac{a^2+b^2-c^2}{2ab}$
$C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$

19. Enrichment: Use your knowledge of the primary trigonometric ratios and the Pythagorean Theorem to prove the Cosine Law:
$c^2 = a^2 + b^2 - 2ab\cos C$



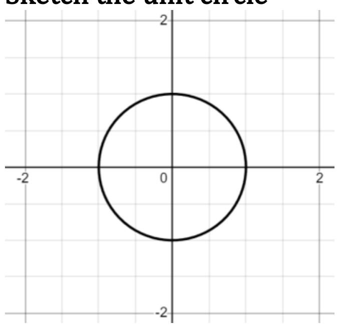
$\sin A = \frac{h}{c} \rightarrow h = c\sin A$
$\cos A = \frac{r}{c} \rightarrow r = c\cos A$
Use the Pythagorean Theorem on the right triangle:
$h^2 + (b - r)^2 = a^2$
$(c\sin A)^2 + (b - c\cos A)^2 = a^2$
$c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A = a^2$
Rearranging we get: $c^2\sin^2 A + c^2\cos^2 A + b^2 - 2bc\cos A = a^2$

Factor: $c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc\cos A = a^2$
Recall that $\sin^2 A + \cos^2 A = 1$
$c^2 + b^2 - 2bc\cos A = a^2$
This is equivalent to $c^2 = a^2 + b^2 - 2ab\cos C$

20. Unit circle:
    a. Equation of the unit circle?
    $x^2 + y^2 = 1$

    b. Enrichment: What is the equation of a circle with radius r centered at the origin?
    $x^2 + y^2 = r^2$

    c. Sketch the unit circle



    d. Explain why $y = \sin\theta$ on the unit circle
    $\sin\theta = \frac{y}{1}$ (1 is the hypotenuse of the triangle)

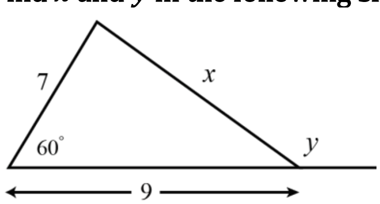
    e. Explain why $x = \cos\theta$ on the unit circle
    $\cos\theta = \frac{x}{1}$ (1 is the hypotenuse of the triangle)

    f. Enrichment: Where does the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ come from?
    Use the Pythagorean Theorem on the unit circle

21. When solving a non-right-angled triangle, when should the Sine Law vs. Cosine Law be used?

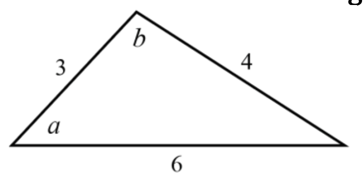
    Use Cosine Law only for SSS and SAS triangles, otherwise use the Sine Law

22. Find x and y in the following SAS triangle:



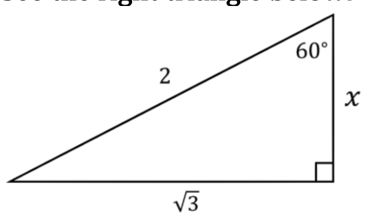
$x^2 = 7^2 + 9^2 - 2(7)(9)\cos 60°$
$x \approx 8.19$
$y \approx 132.2°$

23. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ in the following SSS triangle:



$\mathbf{4^2 = 3^2 + 6^2 - 2(3)(6)\cos a}$
$\boldsymbol{a \approx 36.3^\circ}$
$\boldsymbol{b \approx 117.3^\circ}$

24. See the right triangle below:



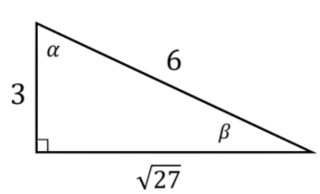
a. Solve $\boldsymbol{x}$ using the Pythagorean Theorem
$\mathbf{x^2 + (\sqrt{3})^2 = 2^2}$
$\mathbf{x^2 + 3 = 4}$
$\mathbf{x^2 = 1}$
$\mathbf{x = 1}$

b. Find the value of the missing angle
$\mathbf{30^\circ}$

c. Solve $\boldsymbol{x}$ using the Sine Law
$\mathbf{\frac{2}{\sin 90^\circ} = \frac{x}{\sin 30^\circ}}$
$\mathbf{x = 1}$

d. Solve $\boldsymbol{x}$ using the Cosine Law
$\mathbf{x^2 = 2^2 + (\sqrt{3})^2 - 2(2)\sqrt{3}\cos 30^\circ}$
$\mathbf{x = \sqrt{4 + 3 - 4\sqrt{3}\cos 30^\circ} = 1}$

25. Solve the unknown angles in the diagram below:



a. Using SOH CAH TOA
$\boldsymbol{\cos \alpha = \frac{3}{6} = \frac{1}{2}}$
$\boldsymbol{\alpha = 30^\circ}$
$\boldsymbol{\beta = 180 - 90 - \alpha = 60^\circ}$

Using similar triangles and your knowledge of a special triangle
This triangle is 3 times larger than the 1-1-$\sqrt{2}$ special triangle which has the same angles.

b. Using the Sine Law

$\frac{\sin\alpha}{\sqrt{27}} = \frac{\sin 90^\circ}{6}$

$\sin\alpha = \frac{1}{6} \times \sqrt{27}$

$\sin\alpha = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

$\alpha = 60^\circ$

$\beta = 180 - 90 - \alpha = 30^\circ$

c. Using the Cosine Law

$\left(\sqrt{27}\right)^2 = 3^2 + 6^2 - 2(3)(6)\cos\alpha$

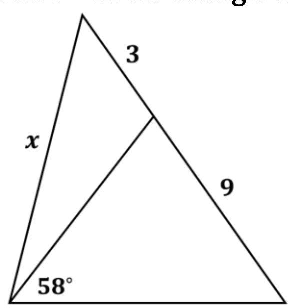
$27 = 9 + 36 - 36\cos\alpha$

$36\cos\alpha = 18$

$\cos\alpha = \frac{18}{36} = \frac{1}{2}$

$\alpha = 60^\circ$

26. Solve x in the triangle below:



SSA – use the Sine Law on the bottom triangle

$\frac{\sin\theta}{10} = \frac{\sin 58^\circ}{9}$

$\sin\theta \approx 0.9422\ldots$

$\theta \approx 70.4^\circ$ (top angle of bottom triangle)

$\frac{y}{\sin 51.56} = \frac{9}{\sin 58}$

$y \approx 8.3127\ldots$ (the unknown side of the top triangle)

SAS – use the Cosine Law on the top triangle

$\alpha = 180 - \theta \approx 109.6^\circ$

$x^2 = 3^2 + y^2 - 2(3)(y)\cos\alpha$

$x \approx 9.74$

27. Basic trigonometric identity: $\tan\theta = \frac{\sin\theta}{?}$

$\cos\theta$

28. Explain how drawing a 2-2-2 equilateral triangle can help you memorize the primary trigonometric ratios.

If you slice the equilateral triangle in half, you get the special 1-2-$\sqrt{3}$ triangle that contain the angles 30°, 60°, and a right angle.

29. Memorize the values of the following special angles:

a. $\sin 30^\circ$

$\frac{1}{2}$

b. $\sin 45^\circ$

$\frac{\sqrt{2}}{2}$

c. $\sin 60^\circ$

$\frac{\sqrt{3}}{2}$

d. $\cos 30^\circ$

$\frac{\sqrt{3}}{2}$

e. $\cos 45^\circ$

$\frac{\sqrt{2}}{2}$

f. $\cos 60^\circ$

$\frac{1}{2}$

g. $\tan 30^\circ$

$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

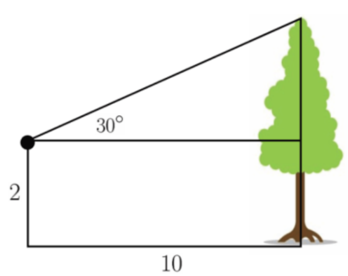
h. $\tan 45^\circ$

1

i. $\tan 60^\circ$

$\sqrt{3}$

30. Find the exact height of the tree without a calculator and simplify your answer using your knowledge of special angles.



$\tan 30^\circ = \frac{x}{10} \rightarrow x = 10\left(\frac{1}{\sqrt{3}}\right)$

$h = \frac{10}{\sqrt{3}} + 2 \text{ or } \frac{10\sqrt{3}}{3} + 2$

31. Evaluate

a. $\sin 120^\circ$

$\frac{\sqrt{3}}{2}$

b. $\cos 330^\circ$

$\frac{\sqrt{3}}{2}$

c. $\sin 225^\circ$

$-\frac{\sqrt{2}}{2}$

d. $-\sin 225^\circ$

$\frac{\sqrt{2}}{2}$

e. $\tan(-420^\circ)$

$-\sqrt{3}$

32. Quadrantal angles – Find:

a. $\sin 90^\circ$

1

b. $\cos 180^\circ$

-1

c. $\sin(-360^\circ)$

0

d. $\tan(180^\circ)$

0

33. Label the (x, y) coordinates on the unit circle for $P(\theta)$ when:

a. $\theta = 30^\circ$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

b. $\theta = 45^\circ$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c. $\theta = 60^\circ$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

d. $\theta = 90^\circ$

$(0, 1)$

e. $\theta = 210^\circ$

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

f. $\theta = 270^\circ$

$(0, -1)$

g. $\theta = 315°$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

h. $\theta = 720°$
$(1, 0)$

34. If $\sin\theta$ is negative and $\cos\theta$ is positive, what quadrant must θ be in?

If $\sin\theta$ is negative then θ must be in Quadrants III or IV.
If $\cos\theta$ is positive then θ must be in Quadrants I or IV.
The only Quadrant that satisfies both conditions is Quadrant IV.

35. Solve the following trigonometric equations within the domain $0 \le \theta \le 360°$:

a. $\sin\theta = \frac{1}{2}$
$\theta_1 = 30°$ and $\theta_2 = 150°$

b. $\sin\theta = -\frac{1}{\sqrt{2}}$
$\theta_1 = 225°$ and $\theta_2 = 315°$

c. $\sin A = \frac{\sqrt{2}}{2}$
$A_1 = 45°$ and $A_2 = 135°$

d. $\sin\beta = -\sqrt{3}/2$
$\beta_1 = 240°$ and $\beta_2 = 300°$

e. $\cos\theta = -0.5$
$\theta_1 = 120°$ and $\theta_2 = 240°$

f. $\tan x = \sqrt{3}$
$x_1 = 60°$ and $x_2 = 240°$

g. $\tan\theta = -2$
$\tan\theta$ is negative in Quadrants II and IV
$\theta_R = \tan^{-1}(2) \approx 63.4°$
$\theta_1 = 180 - \theta_R \approx 116.6°$
$\theta_2 = 360 - \theta_R \approx 296.6°$

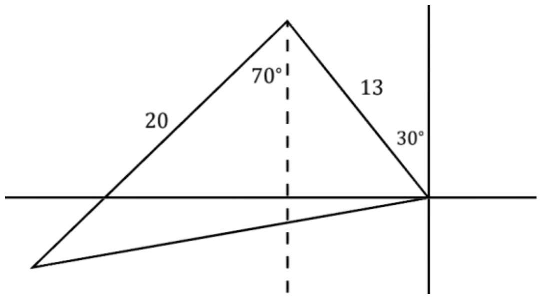
h. $\sin\theta = 2$
No solution. Remember that the range of $\sin\theta$ and $\cos\theta$ is $[-1, 1]$

36. θ in standard position on the unit circle has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Find θ

Recall that on the unit circle: $x = \cos\theta$ and $y = \sin\theta$
Focus on $y = \sin\theta = \frac{1}{2}$
Then $\theta_1 = 30°$ and $\theta_2 = 150°$
But $\cos\theta$ must be negative. So θ must be in Quadrant II.
Thus $\theta = 150°$

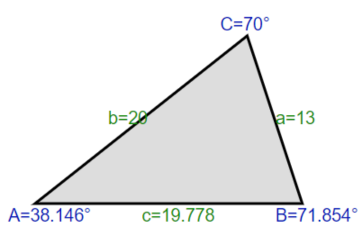
37. Challenge: A boat travels 13 km in the direction $N30°W$.
It then adjusts its course and heads $S70°W$, travelling another 20 km in this new direction.

a. How far is the boat from its initial position?
See diagram below:



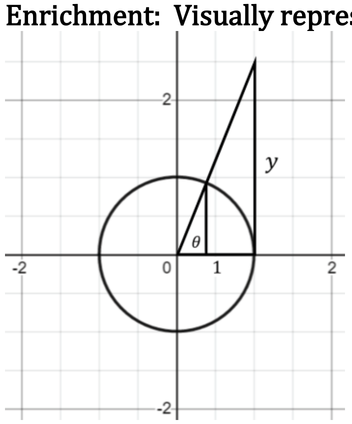
$c^2 = 13^2 + 20^2 - 2(13)(20)\cos(70° + 30°)$ (Cosine Law)
$c \approx 25.7$

b. Enrichment: Bearings are angles measured in a clockwise direction from the north line. What is the bearing of the boat in its final position as compared to its initial position?
Initial bearing is $360° - 30° = 330°$ (heading $N30°W$)
In the final position before the turn the bearing is: $180° + 70° = 250°$
In the final position after the turn the bearing is:



Using a triangle solver we see that $\angle B \approx 71.9°$
The boat begins travelling $N30°W$. $71.9° + 30° = 101.9°$ counter-clockwise. This is equivalent to the bearing of $360 - 101.9° = 258.1°$.

38. Enrichment: Visually represent $\tan\theta$ on the unit circle.



$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{1} \rightarrow y = \tan\theta$ which is the height of the tangent line segment.