Function Transformations: Part I

- Vertical and horizontal translations, stretches, and reflections

1. $f(x) = 2x$. Describe the following transformations:
   a. $y = 3f(x)$
      Multiply y-values by a factor of 3
      $y = 3 \cdot (2x) = 6x$

   b. $y = f(x) + 3$
      Shift 3 up

   c. $y = f(x) - 1$
      Shift 1 down

   d. $y = f(x - 4)$
      Shift 4 right

   e. $y = f\left(x + \frac{3}{2}\right)$
      Shift $\frac{3}{2}$ left

   f. $y = f(x - 1) + 5$
      Shift 1 right 5 up

   g. $y = -\frac{1}{2}f(x)$
      Multiply y's by $-\frac{1}{2}$ (flip about the x-axis)
      $y = -\frac{1}{2} \cdot (2x) = -x$

   h. $y = f(-2x)$
      Multiply x's by $-\frac{1}{2}$ (flip about the y-axis)
      In general when we have $y = f(bx)$ we multiply all x-values by the reciprocal $\frac{1}{b}$.
      When the b value is negative we flip the graph about the x-axis.

2. In general, functions are transformed by the following parameters:
   $g(x) = af\big(b(x \pm c)\big) \pm d$
   Describe the transformational effect of these parameters
   $\times\, y's$ by a
   $\times\, x's$ by $\frac{1}{b}$
   Followed by a horizontal shift of c units
   And a vertical shift of d units

3. $f(x) = x^2$. Describe the following transformations:
   a. $y = f(x) - 4$
      Shift 4 down

   b. $y = f(x) + 2$
      Shift 2 up

   c. $y = f(x - 3)$
      Shift 3 right

d. $y = f(x+1)$
Shift 1 left

e. $y = f(x-2) - 3$
Shift 2 right, 3 down

f. $y = 2f(x)$
$y's \times 2$

g. $y = -f(x)$
Vertical flip about x-axis

h. $y = \frac{1}{2}f(x-2) + 5$
$y's \times \frac{1}{2}$, then shift 2 right, up 5

i. $y = f(2x)$
In general, $y = f(bx)$ means that we multiply the x values by $\frac{1}{b}$
Multiply x-values by a factor of $\frac{1}{2}$

j. $y = f(2x-6)$
Remember to always factor out the coefficient of x
$y = f\big(2(x-3)\big)$
$x's \times \frac{1}{2}$, then shift right 3

k. $y = f(-x)$
Flip horizontally about y-axis

l. $y = \frac{2}{3}f\left(\frac{x}{2} + 4\right) - 1$
$y = \frac{2}{3}f\left(\frac{1}{2}(x+8)\right) - 1$
$y's \times \frac{2}{3}$, $x's \times 2$, followed by a shift 8 left and 1 down

m. $y = -2f\left(4 - \frac{3x}{4}\right) + 3$
$y = -2f\left(-\frac{3}{4}\left(x - \frac{4}{3} \times 4\right)\right) + 3$
$y = -2f\left(-\frac{3}{4}\left(x - \frac{16}{3}\right)\right) + 3$
$y's \times -2$, $x's \times -\frac{4}{3}$, followed by a shift $\frac{16}{3}$ units to the right and 3 units up

4. $f(x) = 2x + 3$. Given $g(x) = f(x) - 2$, find the actual equation of $g(x)$
$g(x) = (2x+3) - 2 = 2x + 1$

5. $f(x) = x^2$. $g(x) = -2f(x+1) - 3$. Find the actual equation of $g(x)$
$g(x) = -2 \cdot (x+1)^2 - 3$

6. $f(x) = x^2 + 2x$. $h(x) = \frac{1}{2}f(x-1)$. Find the actual equation of $h(x)$
$h(x) = \frac{1}{2} \cdot [(x-1)^2 + 2(x-1)]$ or $\frac{1}{2}(x-1)^2 + (x-1)$ or expand fully.

7. $f(x) = 2x^2 - 3x + 1$. $g(x) = -2f(2x - 6) + 3$.
Find the actual equation of $g(x)$.

8. $f(x) = \log_2(x + 2) - \frac{3^x}{e^{\sin x}}$. $g(x) = f(x - 5)$. Find the actual equation of $g(x)$.

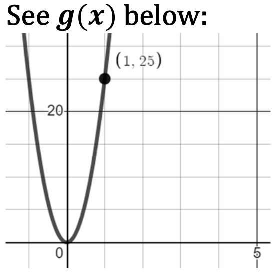
9. $h(t) = 3t^2 - t + 1$. $g(t) = -2h(1 - 3t) - h(2t) + 5$.
   a. Find the actual question of $g(t)$.

   b. Evaluate $h(-1)$

10. Classic function substitution problem: $f(x) = x^2$. $g(x) = \frac{f(x+h)-f(x)}{(x+h)-x}$.
Actual equation of $g(x)$?
$g(x) = \frac{(x+h)^2-x^2}{h} = \frac{x^2+2hx+h^2-x^2}{h} = \frac{h(2x+h)}{h} = 2x + h$

11. $f(x) = x^2$. $g(x) = 9x^2$. $h(x) = f(3x)$. Show that $h(x) = g(x)$.
$h(x) = f(3x) = (3x)^2 = 9x^2$

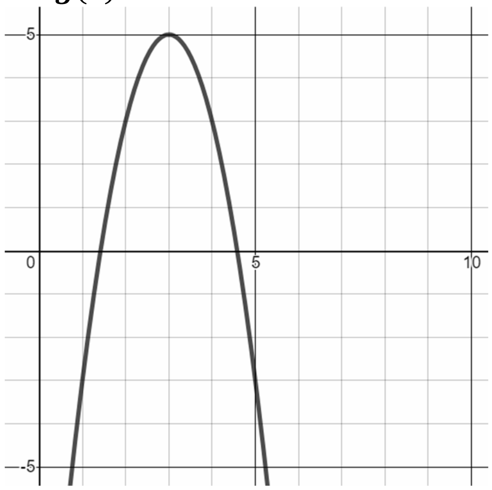
12. See $g(x)$ below:



   a. Describe $g(x)$ as a vertical transformation of $f(x) = x^2$
   $y = ax^2$
   Substitute point $(1, 25)$
   $25 = a(1)^2$
   $25 = a$
   Thus $g(x) = 25 \times f(x)$

   b. Describe $g(x)$ as a horizontal transformation of $f(x) = x^2$
   $g(x) = f(bx)$
   $25x^2 = f(bx)$
   $25x^2 = (bx)^2$
   $25x^2 = (5x)^2$
   $b = 5$ which means that we multiply each x-value by $\frac{1}{5}$.
   In other words, $g(x)$ is 5 times skinnier than the original graph $f(x)$.

13. See $g(x)$ below



a. What is the equation of $g(x)$?

$g(x) = a(x-b)^2 + c$

$y = a(x-3)^2 + 5$

Substitute point $(1, -3)$

$-3 = a(1-3)^2 + 5$

$-8 = 4a$

$a = -2$

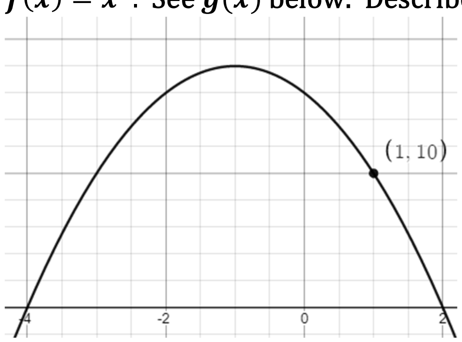
$g(x) = -2(x-3)^2 + 5$

b. Describe $g(x)$ as a transformation of $f(x) = x^2$

$g(x) = -2f(x-3) + 5$

In other words, $\times$ y's by -2, shift 3 right and 5 up

14. $f(x) = x^2$. See $g(x)$ below. Describe $g(x)$ as a transformation of $f(x)$.



$g(x) = a(x+b)(x+c)$

$g(x) = a(x+4)(x-2)$

Substitute the point $(1, 10)$

$10 = a(1+4)(1-2)$

$10 = -5a \rightarrow a = -2$

$g(x) = -2(x+4)(x-2) = -2(x^2 + 2x - 8)$

Now let's complete the square on $x^2 + 2x - 8$

$x^2 + 2x - 8 = (x+1)^2 - 1 - 8 = (x+1)^2 - 9$

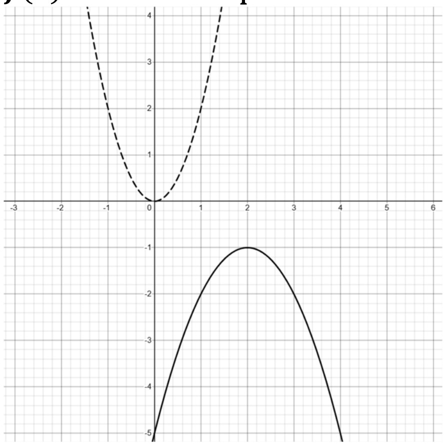
We now have: $-2[(x+1)^2 - 9] = -2(x+1)^2 + 18$

Thus the vertex is $(-1, 18)$

I.e. $g(x) = -2f(x+1) + 18$

Or multiply y's by -2, shift 1 units left and 18 units up

15. $f(x)$ is the dashed parabola below and $g(x)$ is the solid parabola:



a. Find the equation $y = f(x)$.

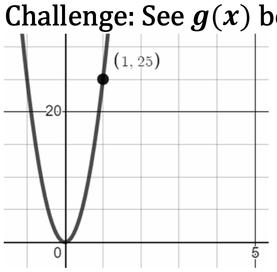
b. Find the equation $y = g(x)$.

c. Describe $g(x)$ as a transformation of $f(x)$.

16. Challenge: $f(x) = x^2$. $g(x) = 5f(x)$. $h(x) = g(x) = f(bx)$. Find the possible values of b.

$g(x) = 5x^2$
$5x^2 = f(bx)$
$5x^2 = (bx)^2$
$5x^2 = b^2x^2$
$5 = b^2$
$b = \pm\sqrt{5}$

17. Challenge: See $g(x)$ below:



$f(x) = x^2$. See $g(x)$ in the diagram above.
Now describe $g(x)$ as a combination of both a vertical and horizontal transformation of $f(x)$.

Previously we found that $g(x) = 25f(x)$
Suppose $g(x) = 5f(kx)$
Then, $g(x) = 5(kx)^2 = 5k^2x^2$
Equating coefficients, $25 = 5k^2$
$k^2 = 5$
$k = \sqrt{5}$
So one possible solution is $g(x) = 5f(\sqrt{5}x)$
Describing this transformation:
$\times$ $y's$ by 5 and $\times$ $x's$ by $\frac{1}{\sqrt{5}}$ which is equivalent to $25f(x)$ and $f(5x)$