CA12 Functions and Graphs Lesson

- Parent functions from Pre-Calculus 12 (ex. exponential, logarithmic, polynomial, rational, trigonometric)
- Piecewise functions
- Inverse trigonometric functions

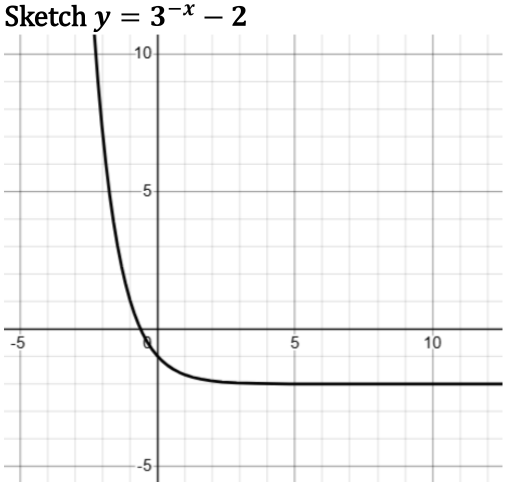
1. Calculus is the study of continuous c__________, and was developed independently in the late 17th century by N_______ and L_________.
   change
   Newton
   Leibniz

2. Another word for instantaneous slope is rate of c______________.
   change

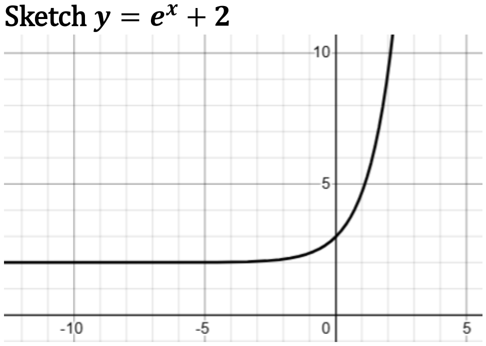
3. Why do so many fields of study require the study of Calculus?
   Fields like physics, engineering, economics, biology, computer science, and even medicine require calculus because it's a universal language for quantifying and solving problems involving change and complexity.

4. Where does Calculus fit on the "map of mathematics"?
   Calculus is a foundational branch of mathematical analysis, which studies continuous change and functions. It includes differential calculus (rates of change, derivatives) and integral calculus (accumulation, integrals). On the map, analysis sits alongside other branches like algebra, geometry, topology, and number theory in pure mathematics.

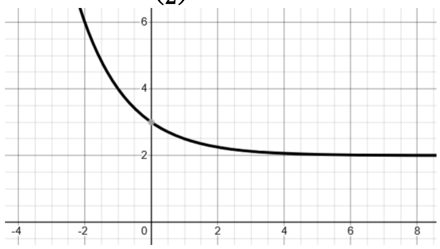
5. Sketch $y = 3^{-x} - 2$



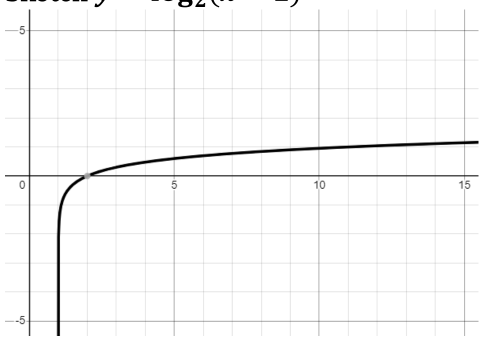
6. Sketch $y = e^{x} + 2$



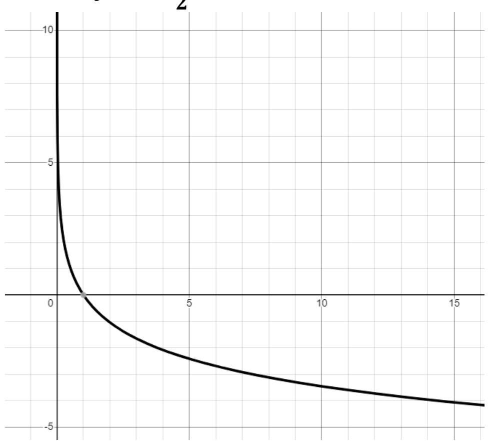
7. Sketch $y = \left(\frac{1}{2}\right)^x + 2$



8. Sketch $y = \log_2(x - 1)$



9. Sketch $y = \frac{-\ln x^3}{2}$



10. Sketch $y = x^3 - 3x^2 + 4$

We use the factor theorem:

Try all the factors of the constant term 4.

Try $P(1), P(-1), P(2), P(-2), P(4), P(-4)$

$P(2) = (2)^3 - 3(2)^2 + 4 = 0$

Thus $x = 2$ is a root and $(x - 2)$ is a factor of the polynomial.

Using long division: $\frac{x^3-3x^2+4}{x-2} = (x - 2)(x + 1)$

Thus $x^3 - 3x^2 + 4 = (x - 2)^2(x + 1)$

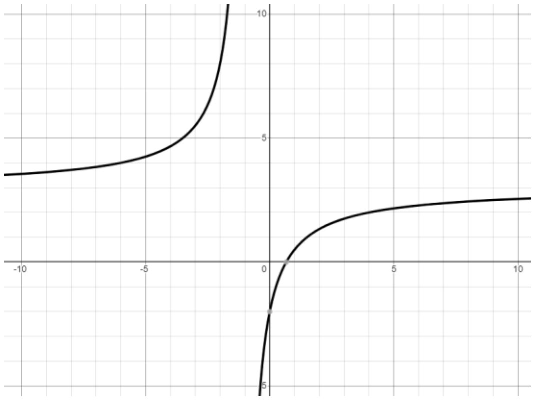
Notice that the multiplicity of the polynomial is 2 at $x = 2$ and is 1 at $x = -1$.

The overall shape of the graph follows the term with the highest power: x^3.

11. Sketch $y = \frac{3x-2}{x+1}$

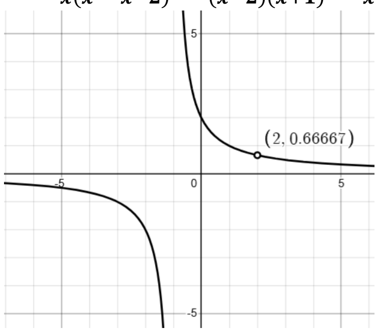
$x = -1$

$y = \frac{3}{1}$



12. Sketch $y = \frac{2x^2-4x}{x^3-x^2-2x}$

$y = \frac{2x(x-2)}{x(x^2-x-2)} = \frac{2(x-2)}{(x-2)(x+1)} = \frac{2}{x+1}$



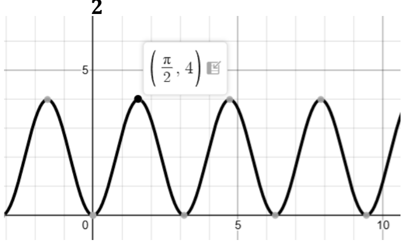
13. Sketch $y = -2\cos 2x + 2$

$\cos(bx)$

The b-value affect the horizontal compression

Per $= \frac{2\pi}{b}$

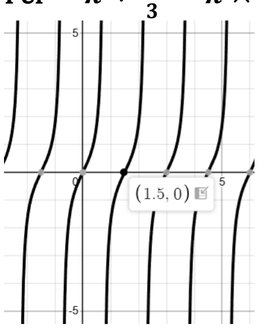
Per $= \frac{2\pi}{2} = \pi$



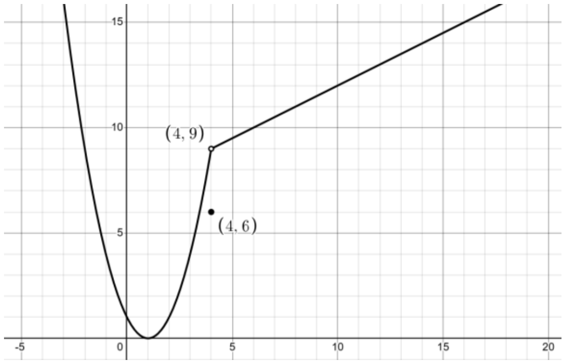
14. Sketch $y = \tan\left(\frac{2\pi x}{3}\right)$

Per $= \frac{\pi}{b}$

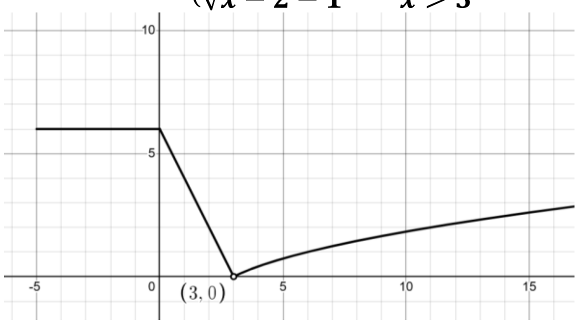
Per $= \pi \div \frac{2\pi}{3} = \pi \times \frac{3}{2\pi} = \frac{3}{2}$



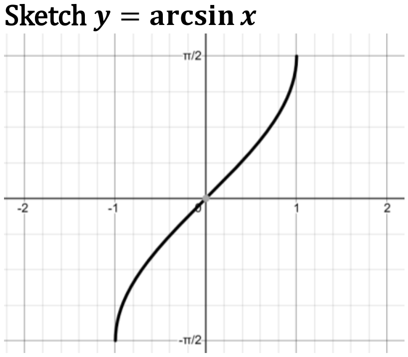
15. Sketch $f(x) = \begin{cases} x^2 - 2x + 1 & x < 4 \\ 6 & x = 4 \\ \frac{x}{2} + 7 & x > 4 \end{cases}$



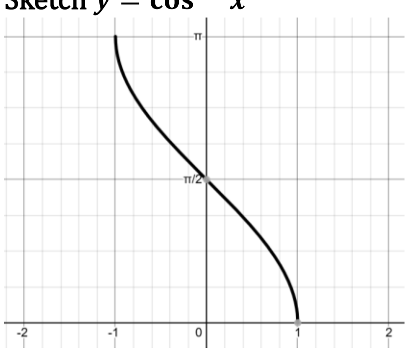
16. Sketch $f(x) = \begin{cases} 6 & x \leq 0 \\ 6 - 2x & 0 < x < 3 \\ \sqrt{x-2} - 1 & x > 3 \end{cases}$



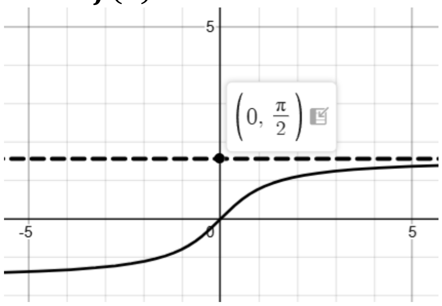
17. Sketch $y = \arcsin x$



18. Sketch $y = \cos^{-1} x$



19. Sketch $f(\theta) = \arctan\theta$



20. $g(x) = \begin{cases} -(x-2)^2 + 4 & x < 3 \\ y = 2x + k & x \geq 3 \end{cases}$

Find the value of k so that $g(x)$ is continuous (defined $x \in \mathbb{R}$)

$-(x-2)^2 + 4 = 2x + k$

$-(x^2 - 4x + 4) + 4 = 2x + k$

$-x^2 + 4x = 2x + k$

$0 = x^2 - 2x + k$

These graph intersect at $x = 3$

$0 = (3)^2 - 2(3) + k$

$0 = 9 - 6 + k$

$-3 = k$

21. Sketch $y = \sqrt{9 - x^2}$

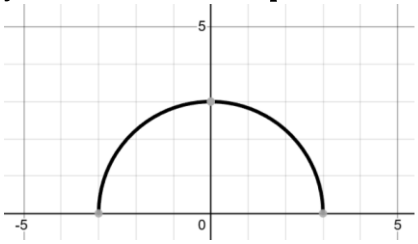
Note: $x^2 + y^2 = 9$ is the equation of a circle with a radius of 3

The general equation of a circle is $x^2 + y^2 = r^2$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9 - x^2}$

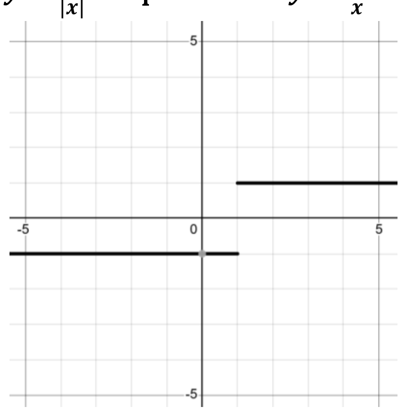
$y = \sqrt{9 - x^2}$ is the top half of a circle



22. Sketch $y = \frac{x-1}{|x-1|}$

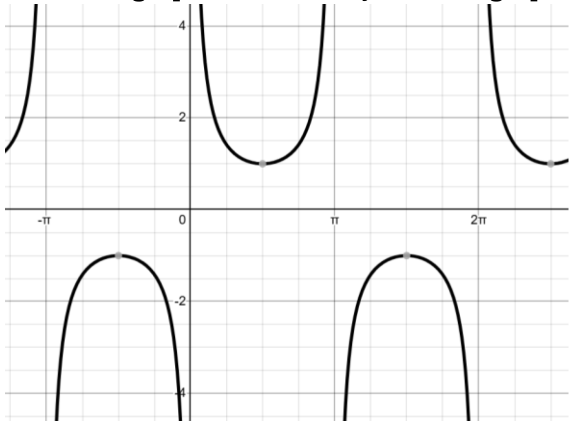
Memorize the $y = \frac{|x|}{x}$ graph.

$y = \frac{x}{|x|}$ is equivalent to $y = \frac{|x|}{x}$. Then shift one unit to the right.

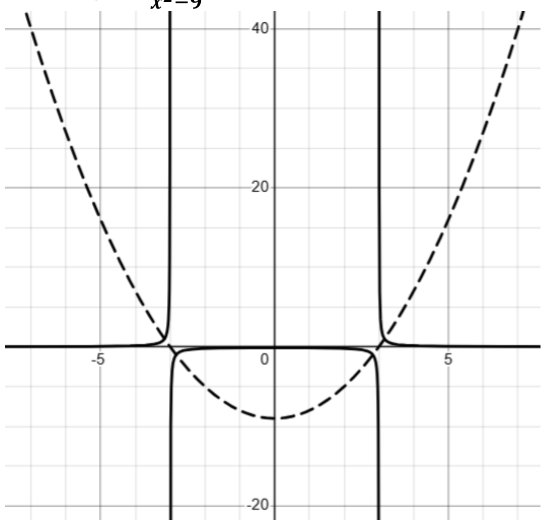


23. Sketch $y = \csc x$

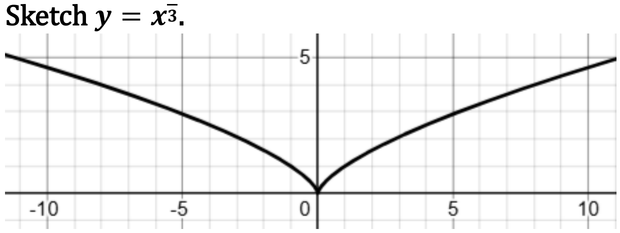
The $\csc x$ graph "kisses" the $y = \sin x$ graph.



24. Sketch $y = \frac{1}{x^2-9}$



25. Sketch $y = x^{\frac{2}{3}}$.



26. Even function, odd function, or neither?

a. $f(x) = x^4 - 2x^2$

$f(x) = f(-x)$ means we have an even function

$f(x) = x^4 - 2x^2$

$f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$

Thus $f(x)$ is an even function.

b. $f(x) = 2x + 3$

$f(x)$ is neither an even or odd function.

c. $f(x) = x^3 + x$

$f(x)$ is an odd function if $f(x) = -f(-x)$

$-f(-x) = -[(-x)^3 + (-x)] = -[-x^3 - x] = x^2 + x = f(x)$

Thus $f(x)$ is an odd function.

d. $f(x) = \tan x + x$
$-f(-x) = -[\tan(-x) + (-x)] = -\tan(-x) + x = \tan x + x$
Thus $f(x)$ is an odd function.

e. $y = \cos^2(2x)$
$\cos(2x)$ is an even function because it is symmetric about the y-axis.

f. $y = e^x + \ln x$
Neither even, nor odd.

g. $y = \frac{1}{x}$
Odd function (symmetric about the origin)

h. $x = y$
$y = 1x + 0$
Odd function
(symmetric about the origin)

Enrichment

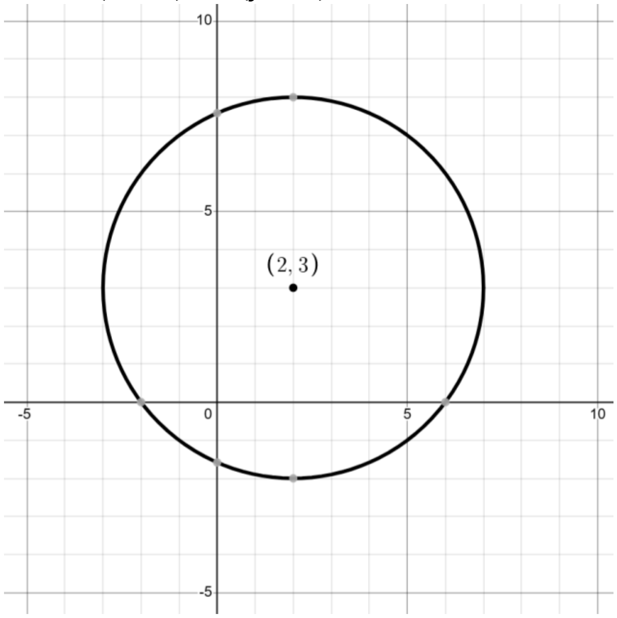
27. Inverse functions:

a. $f(x) = 2^{x-1}$. Find $f^{-1}(x-1)$
$x = 2^{y-1}$
$\log x = \log 2^{y-1}$
$\log x = (y-1)\log 2$
$\frac{\log x}{\log 2} = y - 1$
$\log_2 x + 1 = f^{-1}(x)$
Thus $f^{-1}(x-1) = \log_2(x-1) + 1$

b. $f(x) = (x-2)^2 - 4, x \geq 2$. Find $g(x) = 2f^{-1}(x)$
$x = (y-2)^2 - 4$
$x + 4 = (y-2)^2$
$\pm\sqrt{x+4} = y - 2$
$2 \pm \sqrt{x+4} = f^{-1}(x)$
However $y \geq 2$ on $f^{-1}(x)$ (and $x \geq -3$)
Thus $f^{-1}(x) = 2 + \sqrt{x+4}$
$g(x) = 2\left(2 + \sqrt{x+4}\right) = 4 + 2\sqrt{x+4}$ $(x \geq -3)$

c. $f(x) = x^2 + 4x - 1$. Find $f^{-1}(x)$
$x = y^2 + 4y - 1$
$x + 1 = (y+2)^2 - 4$
$x + 5 = (y+2)^2$
$\pm\sqrt{x+5} = y + 2$
$f^{-1}(x) = \pm\sqrt{x+5} - 2$

28. Sketch $(x-2)^2+(y-3)^2=25$



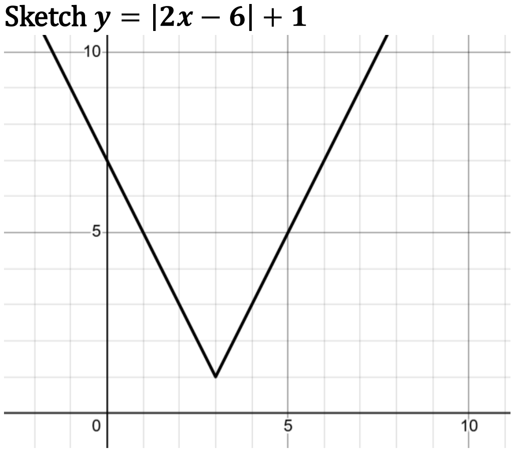
29. Sketch $4x^2+9y^2=36$

Divide each term by 36

$\frac{x^2}{9}+\frac{y^2}{4}=1$

In general the equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse with a horizontal radius of a and a vertical radius of b.

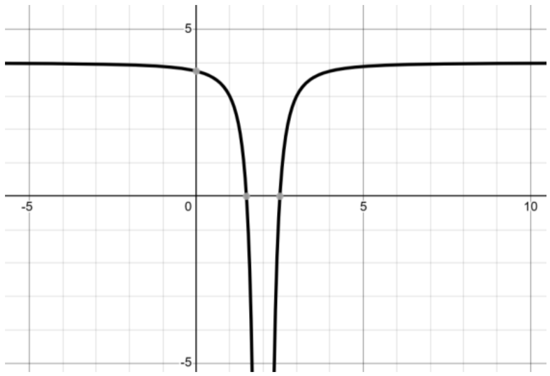
30. Sketch $y=|2x-6|+1$



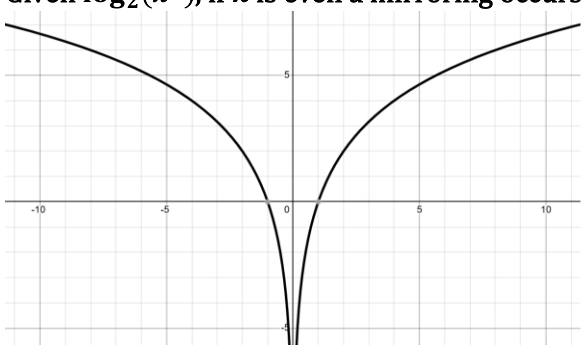
31. Sketch $y=4-\frac{1}{(x-2)^2}$

Learn how to sketch reciprocal functions.

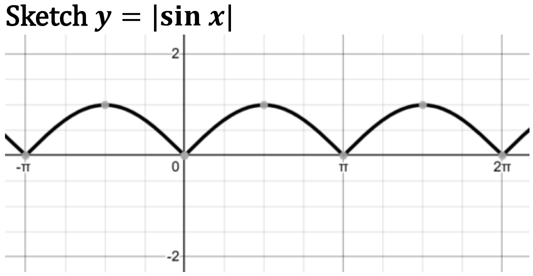
Given $f(x)=(x-2)^2$ can you sketch $y=\frac{1}{f(x)}$?



32. Sketch $y = \log_2 x^2$
Be careful, $\log_2 x^2$ is similar to but not exactly the same as $2\log_2 x$.
Given $\log_2(x^k)$, if k is even a mirroring occurs about the y-axis.

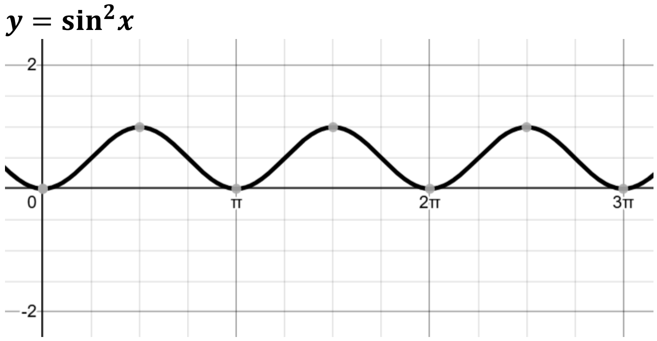


33. Sketch $y = |\sin x|$



34. $f(x) = 1 - \cos^2 x$

a. Sketch
$y = \sin^2 x$

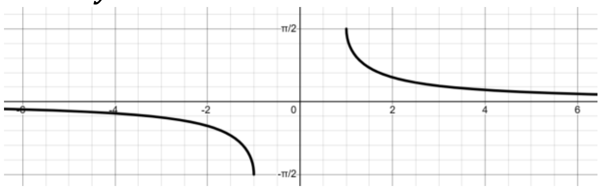


b. Find $g(x) = f(x) = a\cos(2x) + b$
$g(x) = -\frac{1}{2}\cos(2x) + \frac{1}{2}$

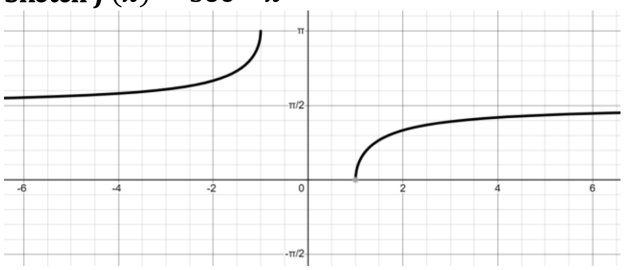
Recall that $\cos(2x) = 1 - 2\sin^2 x$ (Trig. Identities)
Thus $2\sin^2 x = 1 - \cos(2x)$
Dividing both sides by 2:

$\sin^2 x = -\frac{1}{2}\cos(2x) + \frac{1}{2}$

35. Sketch $y = \operatorname{arccsc} x$

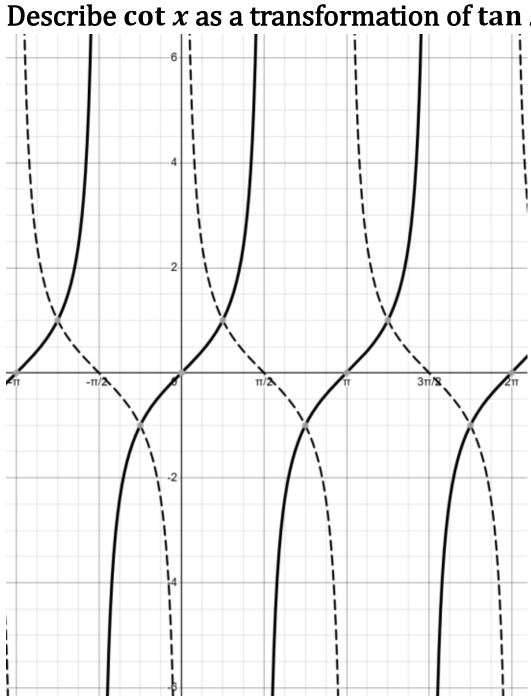


36. Sketch $f(x) = \sec^{-1} x$



37. $f(x) = \tan x$

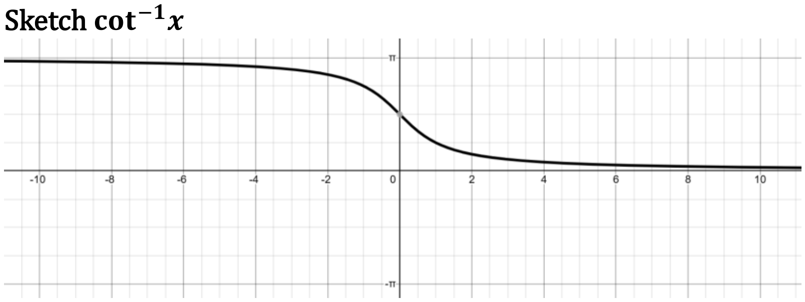
a. Describe $\cot x$ as a transformation of $\tan x$



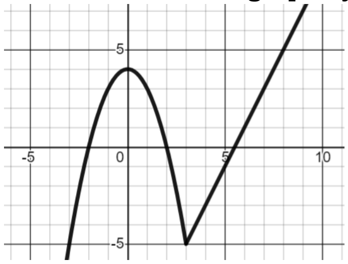
Given $f(x) = \tan x$,

$\cot x = f\left(-\left(x \pm \frac{\pi}{2}\right)\right)$

b. Sketch $\cot^{-1} x$



38. Sketch the piecewise function below using Desmos piecewise notation then transform the graph by flipping it about the x-axis.



Sketch $y = -f(x)$

We can transform the entire piecewise function on Desmos as follows:

$f(x) = \{x \leq 3: -x^2 + 4, x > 3: 2x + 11\}$

$g(x) = -f(x)$