Chain Rule Solutions
(DO NOT WRITE ON THIS PAPER)

1. Given $h(x) = (f \circ g)(x)$ find $h'(x)$ using the chain rule.
   $h(x) = f(g(x))$
   $h'(x) = f'(g(x)) \times g'(x)$

2. The chain rule thats $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. Explain why this seems like a valid equation.
   du is a small value and they cancel out algebraically

3. Use the chain rule to differentiate: $R(x) = \sqrt{5x - 8}$
   $\frac{dR}{dx} = \frac{dR}{du} \times \frac{du}{dx}$
   Let $u = 5x - 8$. Then $R(x) = \sqrt{u} = u^{\frac{1}{2}}$
   Then $\frac{dR}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{5x-8}}$
   $\frac{du}{dx} = \frac{d}{dx}(5x - 8) = 5$
   $\frac{dR}{dx} = \frac{dR}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{5x-8}} \times 5 = \frac{5}{2\sqrt{5x-8}}$

4. Differentiate: $f(x) = \sin(3x^2 + x)$
   $f'(x) = \cos(3x^2 + x) \times (6x + 1)$

5. Differentiate: $f(t) = (2t^3 + \cos(t))^{50}$
   $f'(t) = 50(2t^3 + \cos t)^{49} \times (6t^2 - \sin t)$

6. Differentiate: $h(w) = e^{w^4 - 3w^2 + 9}$
   $\frac{dh}{dw} = e^{w^4 - 3w^2 + 9} \times (4w^3 - 6w)$

7. Differentiate: $g(x) = \ln(x^{-4} + x^4)$
   $g' = \frac{1}{x^{-4} + x^4} \times (-4x^{-5} + 4x^3)$

8. Differentiate: $y = \sec(1 - 5x)$
   Recall that $\frac{d}{dx}\sec u = \sec u \tan u \times \frac{du}{dx}$
   $y' = \sec(1 - 5x)\tan(1 - 5x) \times (-5)$
   $= -5\sec(1 - 5x)\tan(1 - 5x)$

9. Differentiate: $P(t) = \cos^4(t) + \cos(t^4)$
   $P(t) = (\cos t)^4 + \cos(t^4)$
   $P' = 4\cos^3 t\,(-\sin t) - \sin(t^4)(4t^3)$

10. Differentiate: $f(x) = [g(x)]^n$
    $f'(x) = n\, g(x)^{n-1} \times g'(x)$

11. Differentiate: $f(x) = e^{g(x)}$
    $f'(x) = e^{g(x)} \times g'(x)$

12. Differentiate: $f(x) = \ln(g(x))$
    $f' = \frac{1}{g(x)} \times g'(x)$

13. Differentiate: $T(x) = \tan^{-1}(2x)\sqrt[3]{1 - 3x^2}$
    Recall that $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2} \times \frac{du}{dx}$
    Let $f(x) = \tan^{-1}(2x)$ and $g(x) = (1 - 3x^2)^{\frac{1}{3}}$
    $T(x) = f(x) \times g(x)$
    $T' = f'g + g'f$ (product rule)
    $= \frac{2}{1+(2x)^2}(1 - 3x^2)^{\frac{1}{3}} + \frac{1}{3}(1 -$
    $3x^2)^{-\frac{2}{3}}(-6x)\tan^{-1}(2x)$
    Note that the $-6x$ came from the Chain Rule:
    $\frac{d}{dx}(1 - 3x^2) = -6x$
    $= \frac{2\sqrt[3]{1-3x^2}}{1+4x^2} - \frac{6x\tan^{-1}(2x)}{3\sqrt[3]{(1-3x^2)^2}}$

14. Differentiate: $y = \frac{(x^3+4)^5}{(1-2x^2)^3}$
    Let $f(x) = (x^3 + 4)^5$ and $g(x) = (1 - 2x^2)^3$
    Use the Quotient Rule: $\frac{f'g - g'f}{g^2}$ along with the Chain Rule
    $y' = \frac{5(x^3+4)^4(3x^2)(1-2x^2)^3 - 3(1-2x^2)^2(-4x)(x^3+4)^5}{(1-2x^2)^6}$

15. Differentiate: $h(t) = \left(\frac{2t+3}{6-t^2}\right)^3$
    Combine the Power Rule with the Quotient Rule: $\frac{f'g - g'f}{g^2}$
    Let $f(x) = 2t + 3$ and $g(x) = 6 - t^2$
    $h'(t) = 3\left(\frac{2t+3}{6-t^2}\right)^2 \times \frac{2(6-t^2) - (-2t)(2t+3)}{(6-t^2)^2}$

16. Differentiate: $h(z) = \frac{2}{(4z+e^{-9z})^{10}}$
    $h(z) = 2(4z + e^{-9z})^{-10}$
    $h'(z) = -20(4z + e^{-9z})^{-11} \times (4 + e^{-9z} \times -9)$

17. Differentiate: $f(y) = \sqrt{2y + (3y + 4y^2)^3}$
    $f(y) = \left(2y + (3y + 4y^2)^3\right)^{\frac{1}{2}}$
    Here we use the Chain Rule in a multi-link chain
    $\frac{df}{dy} = \frac{1}{2}\left[2y + (3y + 4y^2)^3\right]^{-\frac{1}{2}} \times \left[2 + 3(3y +\right.$

$4y^2)^2 \times (3+8y)\big]$

18. Differentiate: $y = \tan\left(\sqrt[3]{3x^2} + \ln(5x^4)\right)$

    Recall that $\frac{d}{dx}\tan u = \sec^2 u \times \frac{du}{dx}$

    Let $u = \sqrt[3]{3x^2} + \ln(5x^4) = (3x^2)^{\frac{1}{3}} + \ln(5x^4) = \sqrt[3]{3}x^{\frac{2}{3}} + \ln(5x^4)$

    $y' = \sec^2\left(\sqrt[3]{3x^2} + \ln(5x^4)\right) \times \left(\frac{2\sqrt[3]{3}}{3}x^{-\frac{1}{3}} + \frac{1}{5x^4} \times 20x^3\right)$

    $= \sec^2\left(\sqrt[3]{3x^2} + \ln(5x^4)\right)\left(\frac{2\sqrt[3]{3}}{3\sqrt[3]{x}} + \frac{4}{x}\right)$

    $= \sec^2\left(\sqrt[3]{3x^2} + \ln(5x^4)\right) \times \frac{2\left(\sqrt[3]{3}x^{\frac{2}{3}}+6\right)}{3x}$

19. Differentiate: $g(t) = \sin^3\left(e^{1-t} + 3\sin(6t)\right)$

    $g'(t) = 3\sin^2(e^{1-t} + 3\sin(6t)) \times (e^{1-t}(-1) + 3\cos(6t) \times 6)$

Challenge

20. $h(x) = (f \circ g \circ h)(x)$. Find $h'(x)$

    $h(x) = f\left(g\left(h(x)\right)\right)$

    $h'(x) = f'(g\left(h(x)\right) \times g'\left(h(x)\right) \times h'(x)$

21. Justify why the Chain Rule works

    Let $h(x) = (f \circ g)(x) = f\left(g(x)\right)$.

    The Chain Rule states that $h'(x) = f'\left(g(x)\right) \times g'(x)$

    Using Leibniz's notation, if a variable z depends on the variable y,
    which itself depends on the variable x (i.e. y and z are dependent variables),
    then z, via the intermediate variable of y, depends on x as well.
    The chain rule states:

    $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$

    Algebraically this makes sense as the dy's
    (which are known as a small change in the y values) cancel out.

    Intuition:

    z: position of car

    y: position of bike

    x: position of person walking

    For example, suppose $\frac{dz}{dx}$ represents the change of a car's position relative to the position of a walking person. $\frac{dz}{dy}$ represents the change of a car's position relative to a bike and $\frac{dy}{dx}$ represents the change of a bike's position relative to a person walking. If a car travels twice as fast as a bike and a bike travels four times as fast as a walking person, then the car travels $2 \times 4 = 8$ times as fast as the person.

    $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} = 2 \times 4 = 8$

    Proof:

    Let $y = f(u)$ be a differentiable at u and $u = g(x)$ be differentiable at x. Then, by the definition of derivatives we have:

    $\frac{dy}{dx} = \frac{d}{dx}f\left(g(x)\right) = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$

    $= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} \times \frac{g(x+h)-g(x)}{g(x+h)-g(x)}$

    $= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \times \frac{g(x+h)-g(x)}{h}$

    Notice that $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$ is the definition of $g'(x)$

    Also, $\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}$ is the definition of $f'\left(g(x)\right)$

    To see this more clearly we use a change of variables:

    $t = g(x+h)$ and $u = g(x)$

    $\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \times \frac{g(x+h)-g(x)}{h}$

    $= \lim_{h\to 0}\frac{f(t)-f(u)}{t-u} \times \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

    $= f'(u) \times g'(x)$

    So, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ as expected.