Calculus 12 Topic 4: Introduction to Differentiation

- Power rule, product rule, quotient rule, and chain rule
- Transcendental functions: logarithmic, exponential, trigonometric
- Higher order, implicit
- Applications:
  - Relating graph of $f(x)$ to $f'(x)$ and $f''(x)$
  - Increasing / decreasing, concavity
  - Differentiability, mean value theorem
  - Newton's method
  - Problems in contextual situation including:
    - Optimization
    - Related rates

1. Given $y = x^2$, find the average slope from:
   a. $x = 0$ to $x = 3$

   b. $x = 1$ to $x = 3$

   c. $x = 2$ to $x = 3$

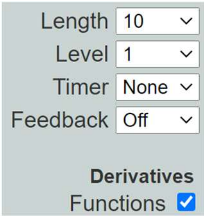
   d. $x = 2.9$ to $x = 3$

   e. $x = 2.99$ to $x = 3$

   f. Can you guess the instantaneous slope at $x = 3$?

2. The Power Rule states that given $f(x) = x^n$, $f'(x)$ or $y' = m = nx^{n-1}$.
   This is the slope formula.
   Given $f(x) = x^2$, use the power rule to find the instantaneous slope at $x = 3$.

3. Use thatquiz.org Calculus to get faster at using the Power Rule:
   Given $y = x^n$, $y' = nx^{n-1}$.
   Check Functions only. What is your personal best time at Level 1?



4. Given $y = 2x^3$, find the slope at $x = -1$

5. Given $f(x) = 2x^2 + 3x$, find $f'(-1)$

6. $y = \sqrt{2}x^{\sqrt{3}}$. Find y'

7. Set $f(x) = x^4$ on www.desmos.com
   a. Try graphing $f'(x)$, $f''(x)$, and so forth ...

   b. Evaluate $f^{(3)}(2)$

   c. Does the "super" notation work on Desmos?

8. $f(x) = x^4 + 2x$. Using the power rule:
   a. Find the equation of the tangent line at $x = 1$.

   b. Find the equation of the normal line at $x = 1$.

9. $f(x) = \sqrt{9 - x^2}$
   a. What is the domain in which this function is continuous?

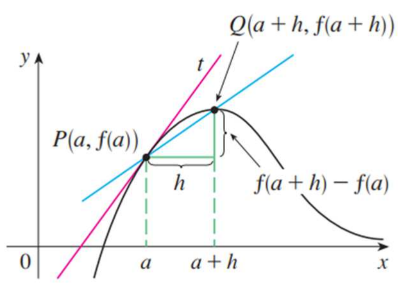
   b. What is the domain in which this function is differentiable?

10. Where is the following function not differentiable?
   a. $f(x) = |2x - 6|$

   b. $f(x) = x^{\frac{2}{3}}$

11. State the limit definition of slope.

12. Explain how this limit definition relates to the diagram below:

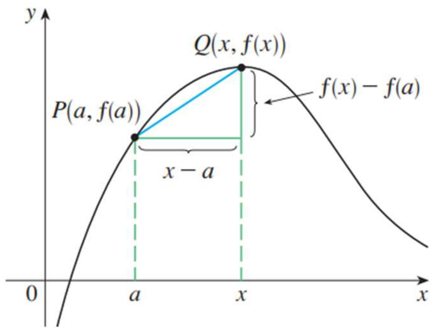


13. $f(x) = x^2$. Find $f'(3)$ using the limit definition of slope.

14. $f(x) = 2x^2 - 3x + 1$. Find $f'(1)$ using the limit definition of slope.

15. State the alternative limit definition of slope.

16. Explain how this alternate limit definition relates to the diagram below:



17. $f(x) = x^2$. Find $f'(3)$ using the alternative limit definition of slope.

18. $f(x) = 2x^2 - 3x + 1$. Find $f'(1)$ using the alternative limit definition of slope.

19. What is the advantage of using:

a. the limit definition of slope formula?

b. the alternate limit definition of slope?

c. the power rule to find the slope?

20. Given $f(x) = \sqrt{2 - x}$ find $f'(1)$ using the limit definition of slope.

21. Given $f(x) = \frac{2}{x}$, find the equation of the tangent line at $x = 1$ using the limit definition of slope.

22. Find $f'(x)$ using the limit definition of slope: $f(x) = \frac{1}{x^2+1}$.

23. Use the alternate definition of slope to find the derivative of $f(x) = x^3$ at $x = 3$.

24. Use the limit definition of slope to prove that

a. given $f(x) = c$ (a constant), $f'(x) = 0$.

b. given $f(x) = |x|$, $f'(0)$ is undefined.

25. Instantaneous slope or rate-of-change can be thought of $m = \frac{\text{rise}}{\text{run}}$ when dealing with a very small triangle. We define $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{?}$

26. Define $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$

27. Given $y = 2x^3$,

a. Find $\frac{dy}{dx}$

b. Find $\frac{d^2y}{dx^2}$

c. Find $\frac{d^3y}{dx^3}$

28. Given $f(x) = \frac{1}{x^2}$, find $\frac{df}{dx}$

29. Evaluate $\frac{d}{dx}\left(\sqrt{5}x^{\sqrt{3}} + \sqrt{2}\right)$

30. Given $f(x) = \sin x$, find $f^{(123)}\left(\frac{\pi}{3}\right)$

31. Memorize common derivative formulas.

Ex. Search for Paul's common derivatives and integrals

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}cx^n = ncx^{n-1}$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}a^x = a^x \ln a$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}\ln |x| = \frac{1}{x}$

$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

32. $f(x) = 2\cos x + \tan x$. Find $f'\left(\frac{\pi}{3}\right)$

33. $f(x) = \arctan x$. Find $f'(1)$

34. $f(x) = e^x - 2^x$. Find the equation of the tangent line at $x = 0$.

35. $f(x) = \ln x + \log_2 x$. Find the equation of the normal line at $x = 1$.

36. Enrichment: Justify why the power rule works: $f(x) = x^n \to f'(x) = nx^{n-1}$.
We know how to factor $x^2 - a^2 = (x+a)(x-a)$ (difference of squares)
$x^3 - a^3 = (x-a)(x^2 + xa + a^2)$ (difference of cubes)
$x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$
$x^5 - a^5 = (x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)$
$x^6 - a^6 = (x-a)(x^5 + x^4a + x^3a^2 + x^2a^3 + xa^4 + a^5)$
Notice the pattern – in general:
$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$
Using the alternate limit definition of slope:
$f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \frac{x^n - a^n}{x-a}$
Using substitution
$= \lim\limits_{x\to a} \frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+x^{n-4}a^3+\cdots+x^2a^{n-3}+xa^{n-2}+a^{n-1})}{x-a}$
$= \lim\limits_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$
$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots aa^{n-1} + a^{n-1}$
Notice that there are n terms
$= na^{n-1}$
Replace a with x
$f'(x) = nx^{n-1}$

37. Enrichment: We can prove many derivative formulas using the limit definition of slope.
Use the limit definition $h \to 0$ to show that $\frac{d}{dx}(\sin x) = \cos x$.
$m = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$
Recall the trigonometric identity: $\sin(A+B) = \sin A\cos B + \cos A \sin B$
Thus $\sin(x+h) = \sin x\cos h + \cos x \sin h$
So $\lim\limits_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x\cos h - \sin x + \cos x \sin h}{h}$
$\lim\limits_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\sin h}{h} = \lim\limits_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim\limits_{h\to 0} \frac{\cos x \sin h}{h}$
$= \sin x \lim\limits_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim\limits_{h\to 0} \frac{\sin h}{h}$

$$= \sin x \lim_{h\to 0} -\frac{1-\cos h}{h} + \cos x \times 1$$

Recall that we proved that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ in the when studying Limits.

We also showed that $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

$$= \sin x \lim_{h\to 0} -(0) + \cos x = 0 + \cos x = \cos x$$

38. $f(x) = e^x$

a. Enrichment: Show that $\frac{d}{dx}e^x = e^x$

Hint: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Let $h = \frac{1}{n}$

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$m = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

Now substitute e

$$\lim_{h\to 0}\frac{e^x[e^h-1]}{h} = \lim_{h\to 0}\frac{e^x\left[\left((1+h)^{\frac{1}{h}}\right)^h - 1\right]}{h} = \lim_{h\to 0}\frac{e^x[1+h-1]}{h} = \lim_{h\to 0}\frac{e^x[1+h-1]}{h} = \lim_{h\to 0} e^x = e^x$$

b. Find the slope at $x = \ln 2$