Calculus 12 Topic 4: Introduction to Differentiation

- Power rule, product rule, quotient rule, and chain rule
- Transcendental functions: logarithmic, exponential, trigonometric
- Higher order, implicit
- Applications:
    - Relating graph of $f(x)$ to $f'(x)$ and $f''(x)$
    - Increasing / decreasing, concavity
    - Differentiability, mean value theorem
    - Newton's method
    - Problems in contextual situation including:
        - Optimization
        - Related rates

1. Given $y = x^2$, find the average slope from:
    a. $x = 0$ to $x = 3$
    When $x = 0, y = 0^2 = 0 \rightarrow (0, 0)$
    When $x = 3, y = 3^2 = 9 \rightarrow (3, 9)$
    $m_{\text{av}} = \frac{9-0}{3-0} = 3$

    b. $x = 1$ to $x = 3$
    When $x = 1, y = 1^2 = 1 \rightarrow (1, 1)$
    When $x = 3, y = 3^2 = 9 \rightarrow (3, 9)$
    $m_{\text{av}} = \frac{9-1}{3-1} = 4$

    c. $x = 2$ to $x = 3$
    When $x = 2, y = 2^2 = 4 \rightarrow (2, 4)$
    When $x = 3, y = 3^2 = 9 \rightarrow (3, 9)$
    $m_{\text{av}} = \frac{9-4}{3-2} = 5$

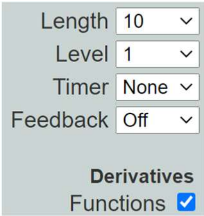
    d. $x = 2.9$ to $x = 3$
    When $x = 2.9, y = 2.9^2 = 8.41 \rightarrow (2.9, 8.41)$
    When $x = 3, y = 3^2 = 9 \rightarrow (3, 9)$
    $m_{\text{av}} = \frac{9-8.41}{3-2.9} = 5.9$

    e. $x = 2.99$ to $x = 3$
    When $x = 2.99, y = 2.99^2 = 8.9401 \rightarrow (2.99, 8.9401)$
    When $x = 3, y = 3^2 = 9 \rightarrow (3, 9)$
    $m_{\text{av}} = \frac{9-8.9401}{3-2.99} = 5.99$

    f. Can you guess the instantaneous slope at $x = 3$?
    6

2. The Power Rule states that given $f(x) = x^n$, $f'(x)$ or $y' = m = nx^{n-1}$.
This is the slope formula.
Given $f(x) = x^2$, use the power rule to find the instantaneous slope at $x = 3$.
$f'(x) = 2x = 2(3) = 6$

3. Use thatquiz.org Calculus to get faster at using the Power Rule:
   Given $y = x^n$, $y' = nx^{n-1}$.
   Check Functions only. What is your personal best time at Level 1?



4. Given $y = 2x^3$, find the slope at $x = -1$
   $y' = 6x^2 = m = 6(-1)^2 = 6$

5. Given $f(x) = 2x^2 + 3x$, find $f'(-1)$
   $f'(x) = 4x + 3$
   $f'(-1) = 4(-1) + 3 = -4 + 3 = -1$

6. $y = \sqrt{2}x^{\sqrt{3}}$. Find y'
   $y' = \sqrt{6}x^{\sqrt{3}-1}$

7. Set $f(x) = x^4$ on www.desmos.com
   a. Try graphing $f'(x), f''(x)$, and so forth ...
      $f'(x) = 4x^3$
      $f''(x) = 12x^2$
      $f'''(x) = 24x$
      $f''''(x) = 24$
      $f'''''(x) = 0$

   b. Evaluate $f^{(3)}(2)$
      $f^{(3)}(2) = f'''(2) = 24(2) = 48$

   c. Does the "super" notation work on Desmos?
      No

8. $f(x) = x^4 + 2x$. Using the power rule:
   a. Find the equation of the tangent line at $x = 1$.
      $f'(x) = 4x^3 + 2$
      $f'(1) = 4(1)^3 + 2 = 6$
      When $x = 1, y = (1)^4 + 2(1) = 3 \rightarrow (1, 3)$
      $y - y_1 = m(x - x_1)$
      $y - 3 = 6(x - 1)$

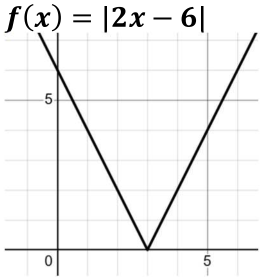
   b. Find the equation of the normal line at $x = 1$.
      $m_{\perp} = -\frac{1}{6}$
      $y - 3 = -\frac{1}{6}(x - 1)$

9. $f(x) = \sqrt{9 - x^2}$
   a. What is the domain in which this function is continuous?
      $x \in [-3, 3]$

b. What is the domain in which this function is differentiable?
$\boldsymbol{x \in (-3, 3)}$

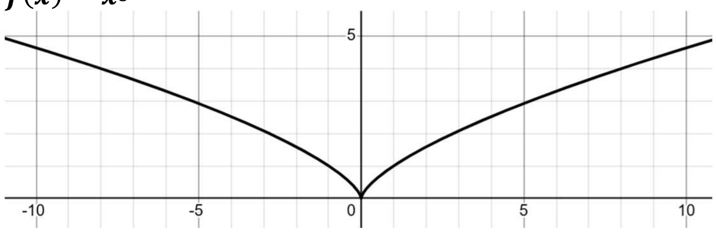
10. Where is the following function not differentiable?

a. $\boldsymbol{f(x) = |2x - 6|}$



Not differentiable at $\boldsymbol{x = 3}$ (corner or cusp)

b. $\boldsymbol{f(x) = x^{\frac{2}{3}}}$

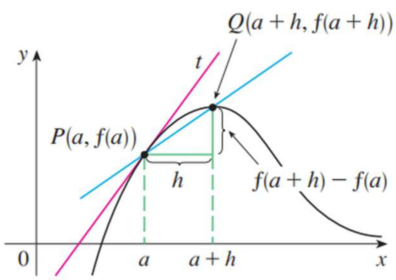


Not diffentiable at $\boldsymbol{x = 0}$

11. State the limit definition of slope.

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

12. Explain how this limit definition relates to the diagram below:



From Stewart Calculus 7th p106

Notice in the diagram $\boldsymbol{x = a}$

$$\text{Slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

As $\boldsymbol{h \to 0}$ we find the instantaneous slope (also known as rate of change) at $\boldsymbol{x = a}$.

Thus $\boldsymbol{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$ or $\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$.

13. $\boldsymbol{f(x) = x^2}$. Find $\boldsymbol{f'(3)}$ using the limit definition of slope.

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h}}$$

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0}(2x + h) = 2x + 0 = 2x}$$

Thus $\boldsymbol{f'(3) = 2(3) = 6}$

14. $\boldsymbol{f(x) = 2x^2 - 3x + 1}$. Find $\boldsymbol{f'(1)}$ using the limit definition of slope.

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) + 1 - [2x^2 - 3x + 1]}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}}$$

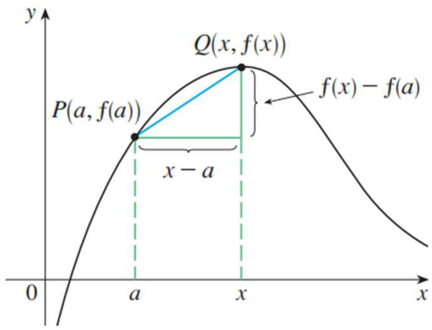
$$\boldsymbol{= \lim_{h \to 0} \frac{h(4x + 2h - 3)}{h} = 4x + 2(0) - 3 = 4x - 3}$$

$f'(1) = 4(1) - 3 = 1$

15. State the alternative limit definition of slope.

$f'(x) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

16. Explain how this alternate limit definition relates to the diagram below:



From Stewart Calculus 7th p104

The slope of the triangle is defined as $\frac{\text{rise}}{\text{run}} = \frac{f(x)-f(a)}{x-a}$

As $x \to a$ we get the instantaneous slope at x i.e $f'(x)$.

17. $f(x) = x^2$. Find $f'(3)$ using the alternative limit definition of slope.

$f'(x) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to 3} \frac{x^2-(3)^2}{x-3} = \lim\limits_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = 3+3 = 6$

18. $f(x) = 2x^2 - 3x + 1$. Find $f'(1)$ using the alternative limit definition of slope.

$f'(x) = \lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1} \frac{2x^2-3x+1-[2(1)^2-3(1)+1]}{x-1} = \lim\limits_{x \to 1} \frac{2x^2-3x+1}{x-1} = \lim\limits_{x \to 1} \frac{(2x-1)(x-1)}{(x-1)}$

$= 2(1) - 1 = 1$

19. What is the advantage of using:
   a. the limit definition of slope formula?
      Because you have the general formula for slope, you can quickly calculate different slopes of a graph.
   b. the alternate limit definition of slope?
      You can quickly find the slope at a point.
   c. the power rule to find the slope?
      The power rule is faster than the limit definition of slope.

20. Given $f(x) = \sqrt{2-x}$ find $f'(1)$ using the limit definition of slope.

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h} \times \frac{\left(\sqrt{2-(x+h)}+\sqrt{2-x}\right)}{\left(\sqrt{2-(x+h)}+\sqrt{2-x}\right)} = \lim\limits_{h \to 0} \frac{2-(x+h)-(2-x)}{h\left(\sqrt{2-(x+h)}+\sqrt{2-x}\right)}$

$= \lim\limits_{h \to 0} \frac{-h}{h\left(\sqrt{2-(x+h)}+\sqrt{2-x}\right)} = \lim\limits_{h \to 0} \frac{-1}{\left(\sqrt{2-(x+h)}+\sqrt{2-x}\right)} = \frac{-1}{\sqrt{2-(1+0)}+\sqrt{1}} = \frac{-1}{2}$

21. Given $f(x) = \frac{2}{x}$, find the equation of the tangent line at $x = 1$ using the limit definition of slope.

When $x = 1, y = \frac{2}{1} = 2 \to (1, 2)$

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{\frac{2}{x+h}-\frac{2}{x}}{h} = \lim\limits_{h \to 0} \frac{1}{h}\left(\frac{2}{x+h} - \frac{2}{x}\right) = \lim\limits_{h \to 0} \frac{1}{h}\left(\frac{2x}{x(x+h)} - \frac{2(x+h)}{x(x+h)}\right)$

$= \lim\limits_{h \to 0} \frac{1}{h}\left(\frac{-2h}{x(x+h)}\right) = \lim\limits_{h \to 0} \left(\frac{-2}{x(x+h)}\right) = \frac{-2}{x^2}$

$m = -\frac{2}{(1)^2} = -2$

$y - 2 = -2(x - 1)$

22. Find $f'(x)$ using the limit definition of slope: $f(x) = \frac{1}{x^2+1}$.

23. Use the alternate definition of slope to find the derivative of $f(x) = x^3$ at $x = 3$.

24. Use the limit definition of slope to prove that
    a. given $f(x) = c$ (a constant), $f'(x) = 0$.

    b. given $f(x) = |x|$, $f'(0)$ is undefined.

25. Instantaneous slope or rate-of-change can be thought of $m = \frac{\text{rise}}{\text{run}}$ when dealing with a very small triangle. We define $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{?}$
$\frac{dy}{dx}$

26. Define $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$
$\frac{du}{dx}$

27. Given $y = 2x^3$,
    a. Find $\frac{dy}{dx}$
    $\frac{dy}{dx} = 6x^2$

    b. Find $\frac{d^2y}{dx^2}$
    $12x$

    c. Find $\frac{d^3y}{dx^3}$
    12

28. Given $f(x) = \frac{1}{x^2}$, find $\frac{df}{dx}$
$f(x) = x^{-2}$
$\frac{df}{dx} = -2x^{-3} = \frac{-2}{x^3}$

29. Evaluate $\frac{d}{dx}\left(\sqrt{5}x^{\sqrt{3}} + \sqrt{2}\right)$
$\sqrt{15}x^{\sqrt{3}-1}$

30. Given $f(x) = \sin x$, find $f^{(123)}\left(\frac{\pi}{3}\right)$
$= f^{(3)}(x)$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

31. Memorize common derivative formulas.
Ex. Search for Paul's common derivatives and integrals
$\frac{d}{dx}(c) = 0$
$\frac{d}{dx}cx^n = ncx^{n-1}$

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\ln |x| = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

32. $f(x) = 2\cos x + \tan x$. Find $f'\left(\frac{\pi}{3}\right)$

$$f'(x) = -2\sin x + \sec^2 x$$
$$f'\left(\frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{3}\right) + \left(\frac{1}{\cos\frac{\pi}{3}}\right)^2 = -2 \cdot \frac{\sqrt{3}}{2} + \frac{1}{\left(\frac{1}{2}\right)^2} = -\sqrt{3} + 4$$

33. $f(x) = \arctan x$. Find $f'(1)$

$$f(x) = \tan^{-1} x$$
$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+(1)^2} = \frac{1}{2}$$

34. $f(x) = e^x - 2^x$. Find the equation of the tangent line at $x = 0$.

$$f'(x) = e^x - 2^x \ln 2$$
$$f'(0) = e^0 - 2^0 \ln 2 = 1 - \ln 2$$

35. $f(x) = \ln x + \log_2 x$. Find the equation of the normal line at $x = 1$.

$$f'(x) = \frac{1}{x} + \frac{1}{x \ln 2}$$
$$f'(1) = \frac{1}{1} + \frac{1}{1 \cdot \ln 2} = 1 + \frac{1}{\ln 2} = \frac{\ln 2}{\ln 2} + \frac{1}{\ln 2} = \frac{\ln 2 + 1}{\ln 2}$$
$$m_\perp = -\frac{\ln 2}{\ln 2 + 1}$$

When $x = 1, y = \ln 1 + \log_2 1 = 0 \rightarrow (1, 0)$

$$y - 0 = -\frac{\ln 2}{\ln 2 + 1}(x - 1)$$

36. Enrichment: Justify why the power rule works: $f(x) = x^n \rightarrow f'(x) = nx^{n-1}$.

We know how to factor $x^2 - a^2 = (x + a)(x - a)$ (difference of squares)

$x^3 - a^3 = (x-a)(x^2 + xa + a^2)$ (difference of cubes)
$x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$
$x^5 - a^5 = (x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)$
$x^6 - a^6 = (x-a)(x^5 + x^4a + x^3a^2 + x^2a^3 + xa^4 + a^5)$
Notice the pattern – in general:
$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$
Using the alternate limit definition of slope:
$f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \frac{x^n - a^n}{x-a}$
Using substitution
$= \lim\limits_{x\to a} \frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+x^{n-4}a^3+\cdots+x^2a^{n-3}+xa^{n-2}+a^{n-1})}{x-a}$
$= \lim\limits_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$
$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots aa^{n-1} + a^{n-1}$
Notice that there are n terms
$= na^{n-1}$
Replace a with x
$f'(x) = nx^{n-1}$

37. Enrichment: We can prove many derivative formulas using the limit definition of slope. Use the limit definition $h \to 0$ to show that $\frac{d}{dx}(\sin x) = \cos x$.
$m = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$
Recall the trigonometric identity: $\sin(A+B) = \sin A \cos B + \cos A \sin B$
Thus $\sin(x+h) = \sin x \cos h + \cos x \sin h$
So $\lim\limits_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$
$\lim\limits_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \lim\limits_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim\limits_{h\to 0} \frac{\cos x \sin h}{h}$
$= \sin x \lim\limits_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim\limits_{h\to 0} \frac{\sin h}{h}$
$= \sin x \lim\limits_{h\to 0} -\frac{1-\cos h}{h} + \cos x \times 1$
Recall that we proved that $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ in the when studying Limits.
We also showed that $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$
$= \sin x \lim\limits_{h\to 0} -(0) + \cos x = 0 + \cos x = \cos x$

38. $f(x) = e^x$
   a. Enrichment: Show that $\frac{d}{dx} e^x = e^x$
      Hint: $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$
      Let $h = \frac{1}{n}$
      $e = \lim\limits_{h\to 0}(1+h)^{\frac{1}{h}}$
      $m = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim\limits_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim\limits_{h\to 0} \frac{e^x(e^h - 1)}{h}$
      Now substitute e
      $\lim\limits_{h\to 0} \frac{e^x[e^h - 1]}{h} = \lim\limits_{h\to 0} \frac{e^x\left[\left((1+h)^{\frac{1}{h}}\right)^h - 1\right]}{h} = \lim\limits_{h\to 0} \frac{e^x[1+h-1]}{h} = \lim\limits_{h\to 0} \frac{e^x[1+h-1]}{h} = \lim\limits_{h\to 0} e^x = e^x$

   b. Find the slope at $x = \ln 2$
      $f'(x) = e^x$

$$\boldsymbol{m = e^{\ln 2} = 2}$$