CA12 Functions and Graphs Solutions (DO NOT WRITE ON THIS PAPER)

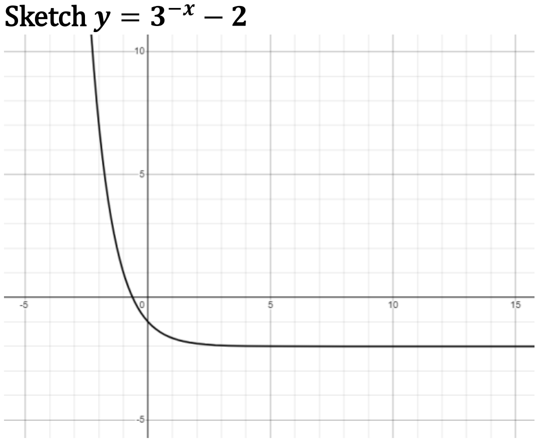
1. Calculus is the study of continuous c___________, and was developed independently in the late 17th century by N_________ and L__________.
   change
   Newton
   Leibniz

2. Another word for instantaneous slope is rate of c________________.
   change

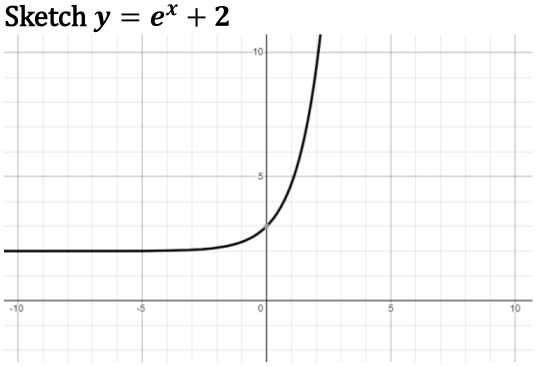
3. Calculus is the mathematical study of change. Give an example of how Calculus is relevant to many
   fields of study such as Biology or Economics.

   Interested in rate-of-change (ex. Population: how fast is the population declining?, economics: how fast money is changing?, Physics: position vs. velocity vs. acceleration)

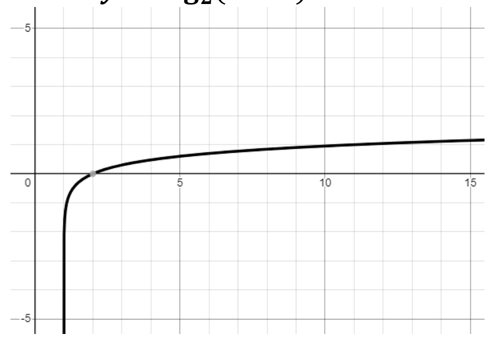
4. Sketch $y = 3^{-x} - 2$



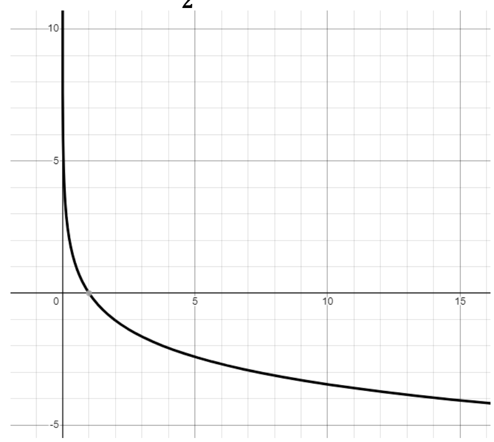
5. Sketch $y = e^x + 2$



6. Sketch $y = \log_2(x-1)$

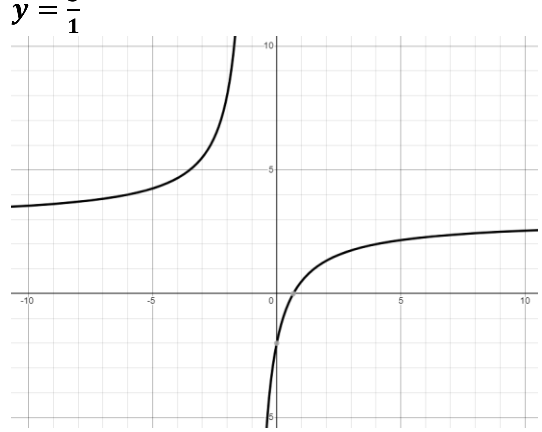


7. Sketch $y = \frac{-\ln x^3}{2}$



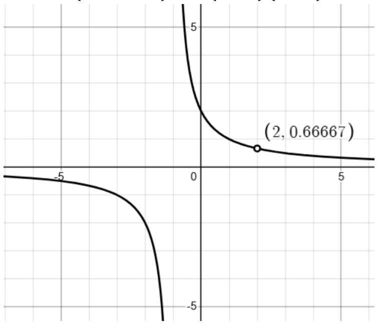
8. Sketch $y = x^3 - 3x^2 + 4$
   $P(2) = 0$
   $(x-2)$ is a factor
   Long or synthetic division
   $y = (x-2)^2(x+1)$

9. Sketch $y = \frac{3x-2}{x+1}$
   $x = -1$
   $y = \frac{3}{1}$



10. Sketch $y = \frac{2x^2-4x}{x^3-x^2-2x}$

$y = \frac{2x(x-2)}{x(x^2-x-2)} = \frac{2(x-2)}{(x-2)(x+1)} = \frac{2}{x+1}$



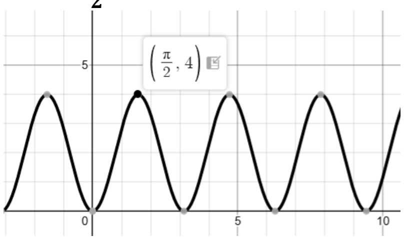
11. Sketch $y = -2\cos 2x + 2$

$\cos(bx)$

The b-value affect the horizontal compression

$\text{Per} = \frac{2\pi}{b}$

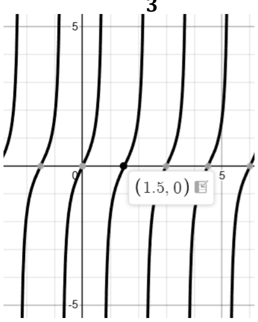
$\text{Per} = \frac{2\pi}{2} = \pi$



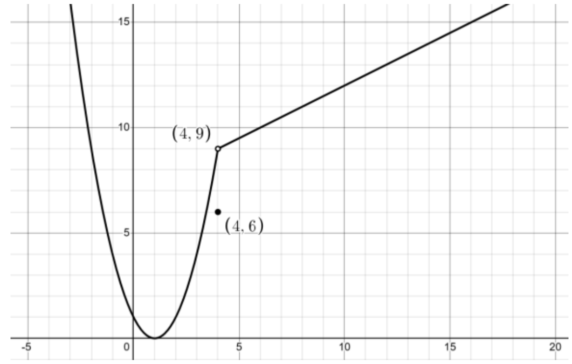
12. Sketch $y = \tan\left(\frac{2\pi x}{3}\right)$

$\text{Per} = \frac{\pi}{b}$

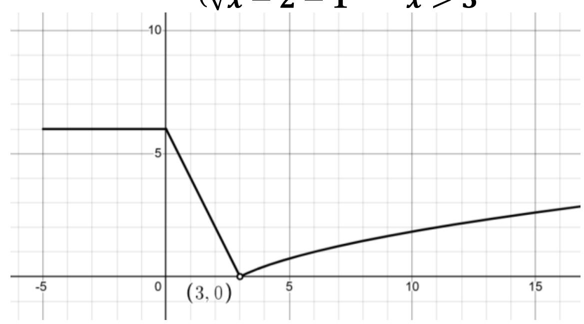
$\text{Per} = \pi \div \frac{2\pi}{3} = \pi \times \frac{3}{2\pi} = \frac{3}{2}$



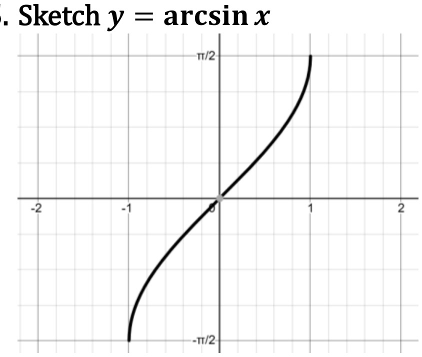
13. Sketch $f(x) = \begin{cases} x^2 - 2x + 1 & x < 4 \\ 6 & x = 4 \\ \frac{x}{2} + 7 & x > 4 \end{cases}$



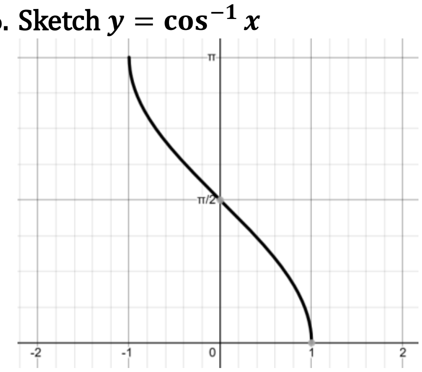
14. Sketch $f(x) = \begin{cases} 6 & x \le 0 \\ 6 - 2x & 0 < x < 3 \\ \sqrt{x-2} - 1 & x > 3 \end{cases}$



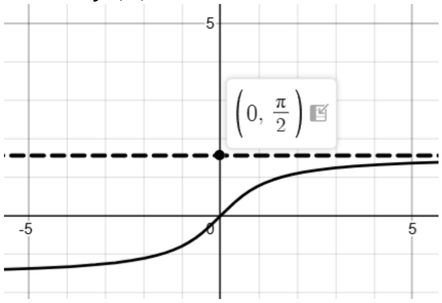
15. Sketch $y = \arcsin x$



16. Sketch $y = \cos^{-1} x$



17. Sketch $f(\theta) = \arctan \theta$



18. $g(x) = \begin{cases} -(x-2)^2 + 4 & x < 3 \\ y = 2x + k & x \geq 3 \end{cases}$

Find the value of k so that $g(x)$ is continuous (defined $x \in \mathbb{R}$)

$-(x-2)^2 + 4 = 2x + k$

$-(x^2 - 4x + 4) + 4 = 2x + k$

$-x^2 + 4x = 2x + k$

$0 = x^2 - 2x + k$

These graph intersect at $x = 3$

$0 = (3)^2 - 2(3) + k$

$0 = 9 - 6 + k$

$-3 = k$

19. Sketch $y = \sqrt{9 - x^2}$

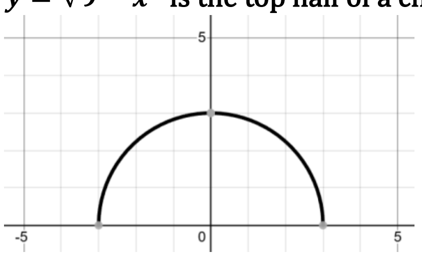
Note: $x^2 + y^2 = 9$ is the equation of a circle with a radius of 3

The general equation of a circle is $x^2 + y^2 = r^2$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9 - x^2}$

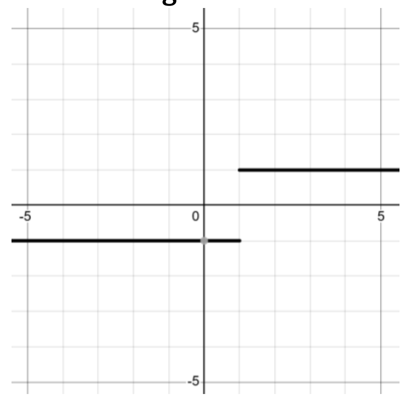
$y = \sqrt{9 - x^2}$ is the top half of a circle



20. Sketch $y = \frac{x-1}{|x-1|}$

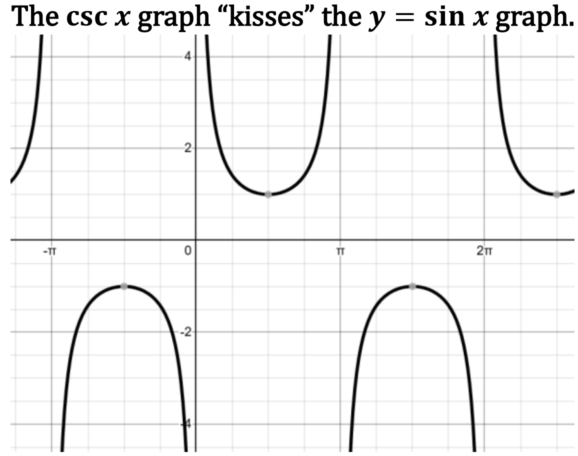
Memorize the $y = \frac{|x|}{x}$ graph.

$y = \frac{x}{|x|}$ is equivalent to $y = \frac{|x|}{x}$. Then shift one unit to the right.



21. Sketch $y = \csc x$

The $\csc x$ graph "kisses" the $y = \sin x$ graph.



22. Even function, odd function, or neither?

a. $f(x) = x^4 - 2x^2$

$f(x) = f(-x)$ means we have an even function

$f(x) = x^4 - 2x^2$

$f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$

Thus $f(x)$ is an even function.

b. $f(x) = 2x + 3$

$f(x)$ is neither an even or odd function.

c. $f(x) = x^3 + x$

$f(x)$ is an odd function if $f(x) = -f(-x)$

$-f(-x) = -[(-x)^3 + (-x)] = -[-x^3 - x] = x^2 + x = f(x)$

Thus $f(x)$ is an odd function.

d. $f(x) = \tan x + x$
$-f(-x) = -[\tan(-x) + (-x)] =$
$-\tan(-x) + x = \tan x + x$
Thus $f(x)$ is an odd function.

e. $y = \cos^2(2x)$
$\cos(2x)$ is an even function because it is symmetric about the y-axis.

f. $y = e^x + \ln x$
Neither even, nor odd.

g. $y = \frac{1}{x}$
Odd function (symmetric about the origin)

h. $x = y$
$y = 1x + 0$
Odd function
(symmetric about the origin)

Challenge

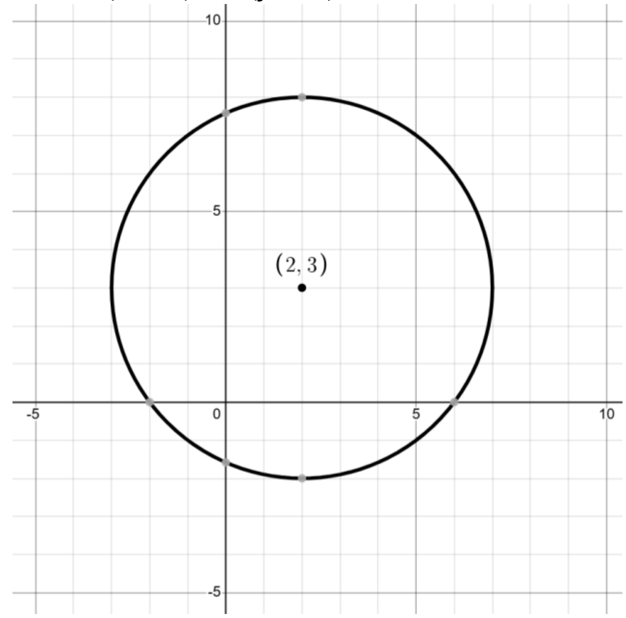
23. Inverse functions:

a. $f(x) = 2^{x-1}$. Find $f^{-1}(x-1)$
$x = 2^{y-1}$
$\log x = \log 2^{y-1}$
$\log x = (y-1)\log 2$
$\frac{\log x}{\log 2} = y - 1$
$\log_2 x + 1 = f^{-1}(x)$
Thus $f^{-1}(x-1) = \log_2(x-1) + 1$

b. $f(x) = (x-2)^2 - 4, x \geq 2$. Find $g(x) = 2f^{-1}(x)$
$x = (y-2)^2 - 4$
$x + 4 = (y-2)^2$
$\pm\sqrt{x+4} = y - 1$
$1 \pm \sqrt{x+4} = f^{-1}(x)$
However $y \geq 2$ on $f^{-1}(x)$ (and $x \geq -3$)
Thus $f^{-1}(x) = 1 + \sqrt{x+4}$
$g(x) = 2\left(1 + \sqrt{x+4}\right) = 2 + 2\sqrt{x+4}$
$(x \geq -3)$

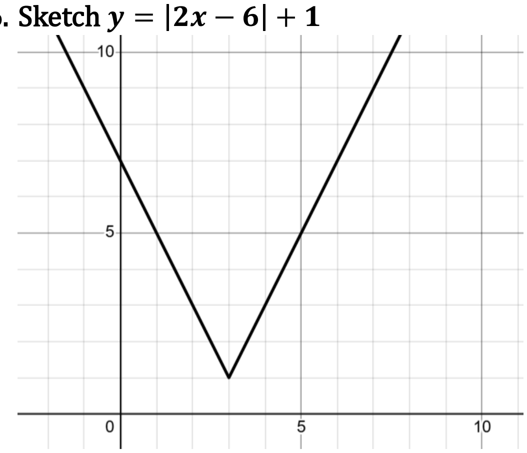
c. $f(x) = x^2 + 4x - 1$. Find $f^{-1}(x)$
$x = y^2 + 4y - 1$
$x + 1 = (y+2)^2 - 4$
$x + 5 = (y+2)^2$
$\pm\sqrt{x+5} = y + 2$
$f^{-1}(x) = \pm\sqrt{x+5} - 2$

24. Sketch $(x-2)^2 + (y-3)^2 = 25$



25. Sketch $4x^2 + 9y^2 = 36$
Divide each term by 36
$\frac{x^2}{9} + \frac{y^2}{4} = 1$
In general the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of an ellipse with a horizontal radius of a and a vertical radius of b.

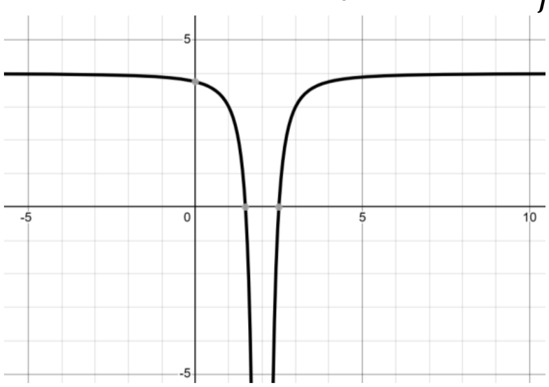
26. Sketch $y = |2x - 6| + 1$



27. Sketch $y = 4 - \frac{1}{(x-2)^2}$

Learn how to sketch reciprocal functions.

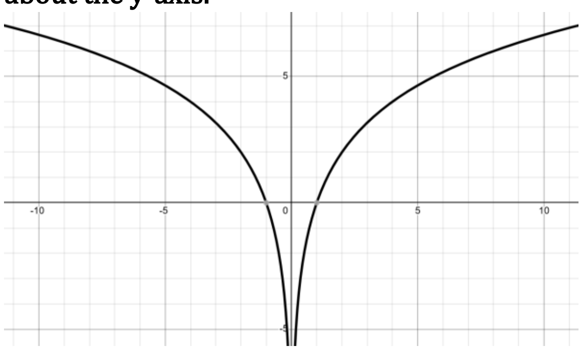
Given $f(x) = (x-2)^2$ can you sketch $y = \frac{1}{f(x)}$?



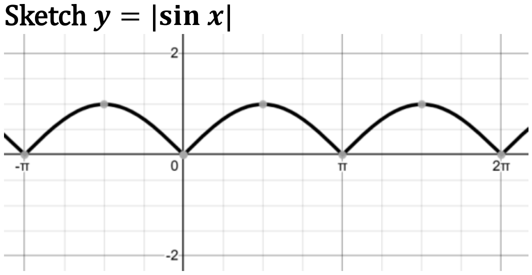
28. Sketch $y = \log_2 x^2$

Be careful, $\log_2 x^2$ is similar to but not exactly the same as $2\log_2 x$.

Given $\log_2(x^k)$, if k is even a mirroring occurs about the y-axis.



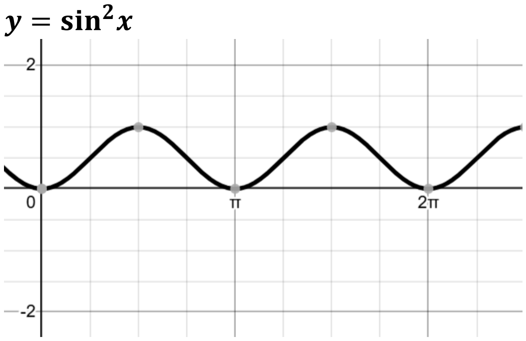
29. Sketch $y = |\sin x|$



30. $f(x) = 1 - \cos^2 x$

a. Sketch

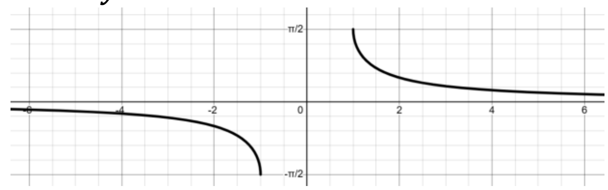
$y = \sin^2 x$



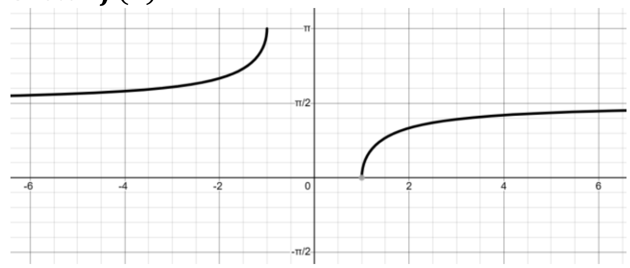
b. Find $g(x) = f(x) = a\cos(2x) + b$

$g(x) = -\frac{1}{2}\cos(2x) + \frac{1}{2}$

31. Sketch $y = \operatorname{arccsc} x$

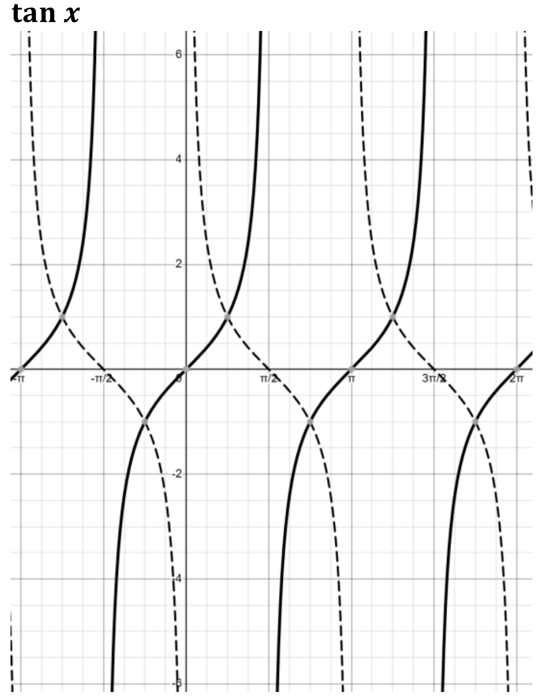


32. Sketch $f(x) = \sec^{-1} x$



33. $f(x) = \tan x$

a. Describe $\cot x$ as a transformation of $\tan x$



Given $f(x) = \tan x$,

$\cot x = f\left(-\left(x \pm \frac{\pi}{2}\right)\right)$

b. Sketch $\cot^{-1} x$