Implicit Differentiation (DO NOT WRITE ON)

Implicit differentiation is important because it allows us to find the derivative of a function when it's not explicitly solved for one variable in terms of another. In many real-world scenarios, relationships between variables are given as equations where isolating one variable isn't straightforward or even possible. With implicit differentiation, we can compute the derivative without having to solve for y explicitly, which can save time and effort – or be the only option when explicit solving isn't feasible.

- Higher order, implicit

1. Find y' for the following function: $x^2 + y^2 = 9$
   In some cases we can isolate for y.
   $y^2 = 9 - x^2$
   $y = \pm\sqrt{9 - x^2}$
   Case 1: $y = \left(9 - x^2\right)^{1/2}$
   $y' = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = -x\left(9 - x^2\right)$
   Case 2: $y = -\left(9 - x^2\right)^{1/2}$
   $y' = -\frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = x\left(9 - x^2\right)^{-\frac{1}{2}}$
   Thus $y' = \pm\frac{x}{\sqrt{9-x^2}}$

   We can also use implicit differentiation - we take the derivative of both sides of the equation:
   $x^2 + y^2 = 9$
   $2x\frac{dx}{dx} + 2y\frac{dy}{dx} = \frac{d}{dx}(9)$
   Same as: $2x + 2yy' = 0$
   The $\frac{dx}{dx}$ and the $\frac{dy}{dx}$ parts exist because of the chain rule
   $2x + 2y\frac{dy}{dx} = 0$
   $2y\frac{dy}{dx} = -2x$
   $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$
   Recall that $y^2 = 9 - x^2 \rightarrow y = \pm\sqrt{9 - x^2}$
   Thus $\frac{dy}{dx} = -\frac{x}{y} = \pm\frac{x}{\sqrt{9-x^2}}$

2. Find the equation of the tangent line to: $x^2 + y^2 = 9$ at the point $\left(2, \sqrt{5}\right)$
   $2x + 2yy' = 0$ (Implicit Differentiation)
   $y' = -\frac{2x}{2y} = -\frac{x}{y} = -\frac{2}{\sqrt{5}}$
   $y - y_1 = m(x - x_1)$
   $y - \sqrt{5} = -\frac{2}{\sqrt{5}}(x - 2)$

3. Given $9x^2 + 16y^2 = 144$, find the equation of the normal line at $x = 3$ given $y > 0$.
   Divide by 144
   $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (ellipse with horizontal radius of 4 and a vertical radius of 3)
   Use implicit differentiation
   $\frac{2}{16}x + \frac{2}{9}yy' = 0$
   Multiply by 72
   $9x + 16yy' = 0$
   $16yy' = -9x$
   $y' = -\frac{9x}{16y}$
   When $x = 3$, $9(3)^2 + 16y^2 = 144$
   $16y^2 = 63$
   $y^2 = \frac{63}{16}$
   Since $y > 0$, $y = \frac{\sqrt{63}}{4}$
   Point $\left(3, \frac{\sqrt{63}}{4}\right)$
   Slope $= -\frac{9x}{16y} = -\frac{9(3)}{16\left(\frac{\sqrt{63}}{4}\right)} = -\frac{9}{4\sqrt{7}}$
   $m_{\perp} = \frac{4\sqrt{7}}{9}$
   $y - y_1 = m(x - x_1)$
   $y - \frac{\sqrt{63}}{4} = \frac{4\sqrt{7}}{9}(x - 3)$

4. Find y': $x^3y^5 + 3x = 8y^3 + 1$
   Use Implicit Differentiation with the Product Rule
   $(fg)' = f'g + g'f$
   $3x^2y^5 + 5y^4y'x^3 + 3 = 24y^2y' + 0$
   $3x^2y^5 + 3 = 24y^2y' - 5y^4y'x^3$
   $3x^2y^5 + 3 = y'\left(24y^2 - 5y^4x^3\right)$
   $y' = \frac{3x^2y^5+3}{24y^2-5x^3y^4}$

5. Find y': $x^2\tan(y) + y^{10}\sec(x) = 2x$
   Use Implicit Differentiation with the Product Rule
   $(fg)' = f'g + g'f$
   Remember that $\frac{d}{dx}\tan y = \sec^2 y \times \frac{dy}{dx}$ and $\frac{d}{dx}\sec x = \sec x \tan x$

$2x \tan y + \sec^2 y \times y'x^2 + 10y^9 y' \sec x +$
$\sec x \tan x \, y^{10} = 2$
$y' = \frac{2 - 2x \tan y - \sec x \tan x \, y^{10}}{\sec^2 y \times x^2 + 10y^9 \times \sec x}$

6. Find y': $e^{2x+3y} = x^2 - \ln(xy^3)$
   Use Implicit Differentiation with the Product Rule
   $(fg)' = f'g + g'f$
   $e^{2x+3y} \times (2 + 3y') = 2x - \frac{1}{xy^3} \times \left((1)y^3 + 3y^2 y' x\right)$
   $2e^{2x+3y} + 3y'e^{2x+3y} = 2x - \frac{y^3 + 3y^2 y' x}{xy^3}$
   Multiply both sides by xy^3
   $2xy^3 e^{2x+3y} + 3xy^3 y' e^{2x+3y} = 2x^2 y^3 - (y^3 + 3y^2 y' x)$
   $2xy^3 e^{2x+3y} + 3xy^3 y' e^{2x+3y} = 2x^2 y^3 - y^3 - 3y^2 y' x$
   $3xy^3 y' e^{2x+3y} + 3y^2 y' x = 2x^2 y^3 - y^3 - 2xy^3 e^{2x+3y}$
   $y'(3xy^3 e^{2x+3y} + 3y^2 x) = 2x^2 y^3 - y^3 - 2xy^3 e^{2x+3y}$
   $y' = \frac{2x^2 y^3 - y^3 - 2xy^3 e^{2x+3y}}{3xy^3 e^{2x+3y} + 3y^2 x}$

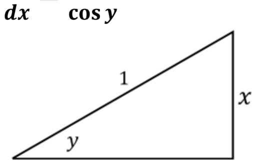
7. Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation
   with respect to t. $x^3 y^6 + e^{1-x} - \cos(5y) = y^2$
   Use Implicit Differentiation with the Product Rule
   $(fg)' = f'g + g'f$
   Because we are differentiating with respect to t and not x the chain
   rule applies to both x and y.
   $3x^2 \times \frac{dx}{dt} \times y^6 + 6y^5 \times \frac{dy}{dt} \times x^3 + e^{1-x} \times (-1) \times \frac{dx}{dt} + \sin(5y) \times 5 \times \frac{dy}{dt} = 2y \times \frac{dy}{dt}$

8. Use your understanding of logarithms and implicit differentiation to show that $\frac{d}{dx} a^x = \frac{a^x}{\ln a}$
   Let $y = a^x$
   Then $\log_a y = x$
   $\frac{\log y}{\log a} = x$
   $\frac{\ln y}{\ln a} = x$
   Differentiate both sides (implicit differentiation)
   $\frac{d}{dx}\left(\frac{\ln y}{\ln a}\right) = \frac{d}{dx}(x)$
   $\frac{1}{\ln a} \times \frac{1}{y} y' = 1$ (the $y' = \frac{dy}{dx}$ came from the chain rule)
   $\frac{1}{\ln a} \times \frac{1}{a^x} \times y' = 1$
   Multiply both sides by the denominator
   $y' = a^x \ln a$

9. Use logarithmic differentiation on $y = x^{2x}$ to find y'
   Log both sides:
   $\ln y = \ln x^{2x}$
   $\ln y = 2x \ln x$
   Use implicit differentiation (along with product rule)
   $\frac{1}{y} y' = 2 \ln x + \frac{1}{x}(2x)$
   Multiply by y:
   $y' = y \times 2 \ln x + 2$
   Note: $\ln y = \ln x^{2x}$
   Thus $e^{\ln x^{2x}} = y$ (now substitute this result):
   $y' = y \times 2 \ln x + 2 = \left(e^{\ln x^{2x}}\right)(2 \ln x + 2) = 2e^{\ln x^{2x}}(\ln x + 1)$

Challenge

10. Justify that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ using implicit differentiation
    $y = \arcsin x = \sin^{-1} x$
    $y = \sin^{-1} x$
    This implies $\sin y = x$
    Use implicit differentiation with respect to x
    $\cos y \frac{dy}{dx} = 1$
    $\frac{dy}{dx} = \frac{1}{\cos y}$



    $\cos y = \frac{\text{adj}}{1}$
    $\text{adj}^2 + x^2 = 1^2$ (Pythagorean Theorem)
    $\text{adj} = \sqrt{1 - x^2}$
    $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

11. Show that $\frac{d}{dx}\sin^{-1}\left(\frac{x}{k}\right) = \frac{1}{\sqrt{k^2-x^2}}$ using implicit differentiation

$y = \sin^{-1}\left(\frac{x}{k}\right)$

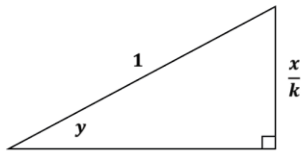
$\sin y = \frac{x}{k}$

$k \sin y = x$

Use implicit differentiation

$k \cos y \, y' = 1$

$y' = \frac{1}{k \cos y}$



Using the Pythagorean Theorem

$?^2 + \left(\frac{x}{k}\right)^2 = 1^2$

$?^2 = 1 - \frac{x^2}{k^2}$

$? = \sqrt{1 - \frac{x^2}{k^2}}$

Thus $\cos y = \sqrt{1 - \frac{x^2}{k^2}}$

$y' = \frac{1}{k\sqrt{1-\frac{x^2}{k^2}}} = \frac{1}{\sqrt{k^2\left(1-\frac{x^2}{k^2}\right)}} = \frac{1}{\sqrt{k^2-x^2}}$

12. Show that $\frac{d}{dx}\frac{1}{k}\tan^{-1}\left(\frac{x}{k}\right) = \frac{1}{k^2+x^2}$

$y = \frac{1}{k}\arctan\left(\frac{x}{k}\right)$

Multiply by k

$ky = \arctan\left(\frac{x}{k}\right)$

tan both sides

$\tan(ky) = \frac{x}{k}$

Use implicit differentiation

$k \sec^2(ky) y' = \frac{1}{k}$

$y' = \frac{1}{k^2 \sec^2(ky)}$

Recall that $\sin^2 x + \cos^2 x = 1$

Divide by $\cos^2 x$

$\tan^2 x + 1 = \sec^2 x$

Thus $y' = \frac{1}{k^2\,(\tan^2(ky)+1)}$

Recall that $\tan(ky) = \frac{x}{k}$

Thus $y' = \frac{1}{k^2\left[\left(\frac{x}{k}\right)^2+1\right]} = \frac{1}{k^2\left[\frac{x^2}{k^2}+1\right]} = \frac{1}{x^2+k^2}$