CA12 Limits and Continuity Lesson

- From a table of values, graphically, and algebraically
- One-sided versus two-sided
- End-behavior
- Intermediate value theorem
- Left and right limits
- Limits to infinity
- Continuity

1. $f(x) = 2x + 3$.
   a. As x approaches very close to $x = 4$, what is the value of $f(x)$?
   $f(x) = 2(4) + 3 = 11$

   b. Write this question as a limit statement:
   $\lim_{x \to 4} (2x + 3) = 11$

2. Find $\lim_{x \to -3} (x^2 + 2x)$
   $(-3)^2 + 2(-3) = 3$

3. Find the limit using substitution:
   a. $f(x) = e^x$. $\lim_{x \to \ln 2} f(x)$
   $= e^{\ln 2} = 2$

   b. $f(x) = \ln x$. Find $\lim_{x \to \left(\frac{1}{e^2}\right)} f(x)$
   $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2 \times \ln e = -2$

   c. $\lim_{x \to \frac{4\pi}{3}} \cos^2 x$
   $\left[\cos\left(\frac{4\pi}{3}\right)\right]^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

   d. $\lim_{x \to -\frac{4\pi}{3}} \tan x$
   $\tan\left(-\frac{4\pi}{3}\right) = -\sqrt{3}$

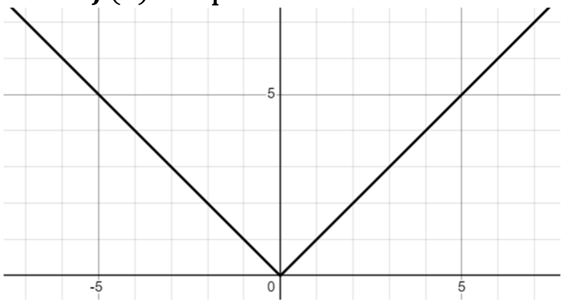
   e. Find $\lim_{x \to 2\pi} \sec x$
   $\sec(2\pi) = \frac{1}{\cos 2\pi} = \frac{1}{1} = 1$

4. If possible, find $\lim_{x \to 3} \frac{x-3}{x^2-9}$
   $\lim_{x \to 3} \frac{(x-3)}{(x+3)(x-3)} = \lim_{x \to 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$

5. If possible, $\lim_{x \to -\frac{3}{2}} \frac{2x+3}{2x^2-5x-12}$
   $\lim_{x \to -\frac{3}{2}} \frac{(2x+3)}{(2x+3)(x-4)} = \lim_{x \to -\frac{3}{2}} \frac{1}{x-4} = \frac{1}{-\frac{3}{2}-4} = -\frac{2}{11}$

6. $f(x) = |x|$

a. Define $f(x)$ as a piecewise function.

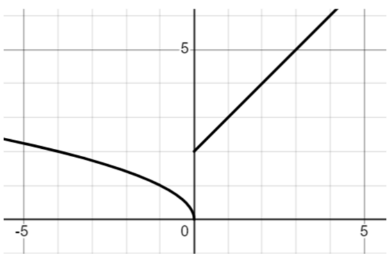


$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

b. If possible, find $\lim_{x \to 0} f(x)$

0

7. See the piecewise function $f(x) = \begin{cases} \sqrt{-x} & x \leq 0 \\ |x| + 2 & x > 0 \end{cases}$ below:



a. Find the left-hand limit $\lim_{x \to 0^-} f(x)$

0

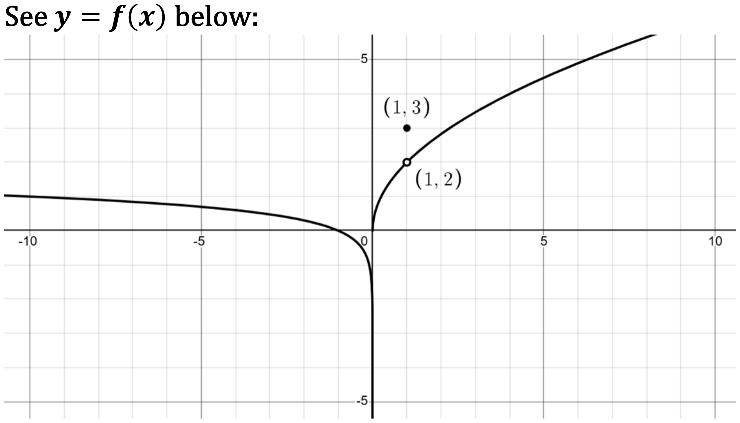
b. Find the right-hand limit $\lim_{x \to 0^+} f(x)$

2

c. If possible, find $\lim_{x \to 0} f(x)$

No limit (left and right hand limits do not exist)

8. See $y = f(x)$ below:



a. Write as a piecewise function

$$f(x) = \begin{cases} \log(-x) & x < 0 \\ \dfrac{2\sqrt{x}(x-1)}{x-1} & x \geq 0, x \neq 1 \\ 3 & x = 1 \end{cases}$$

b. $\lim_{x \to 1} f(x)$

2

c. Evaluate $f(1)$
3

d. $\lim\limits_{x \to 0^-} f(x)$
Undefined or $-\infty$

e. $\lim\limits_{x \to 0^+} f(x)$
0

f. $\lim\limits_{x \to 0} f(x)$
No limit

9. $f(x) = \begin{cases} (x-2)^2 & x > 3 \\ x + k & x < 3 \\ 2 & x = 3 \end{cases}$
Find k so that the limit exists at $x = 3$.
Let $x = 3$
Substitute into $y = (x-2)^2$
$y = (3-2)^2 = 1$
Now substitute $y = 1$ into $y = x + k$
$1 = 3 + k$
$-2 = k$
$k = -2$

10. The definition of a limit: Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that, $\lim\limits_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

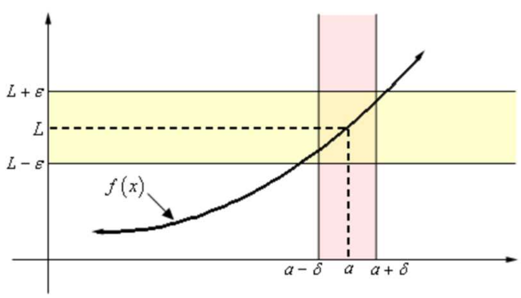


Diagram from https://tutorial.math.lamar.edu/classes/calcI/defnoflimit.aspx

Note: $|x - a| < \delta$
Case 1: $x - a < \delta \to x < a + \delta$
Case 2: $-(x - a) < \delta \to -x + a < \delta \to a - \delta < x$ or $x > a - \delta$

11. $|x - 2| < 5$
a. Solve x expressing the domain as an inequality statement.
$-5 < x - 2 < 5$
Add 2 to each part
$-3 < x < 7$

b. State the domain in plain English.
The distance away from $x = 2$ is within 5 units

c. Now explain the meaning of $|x - a| < \delta$ in plain English.
The distance away from $\boldsymbol{x} = \boldsymbol{a}$ is within $\boldsymbol{\delta}$ units

12. Solve $|2x + 3| \leq 7$
$-7 \leq 2x + 3 \leq 7$
Subtract 3
$-10 \leq 2x \leq 4$
Divide by 2
$-5 \leq x \leq 2$

13. Enrichment:
    a. $|x - 1| \geq 5$
    Case 1: $x - 1 \geq 5 \rightarrow x \geq 6$
    Case 2: $-(x - 1) \geq 5 \rightarrow -x + 1 \geq 5 \rightarrow -4 \geq x$
    Together: $x \leq -4$ or $x \geq 6$

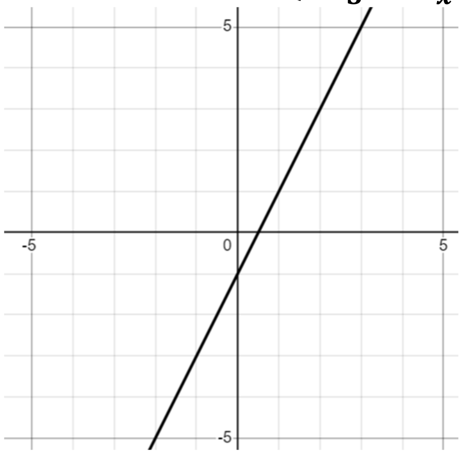
    b. Solve $|3x - 5| > 2$
    Case 1: $3x - 5 > 2$
    $3x > 7$
    $x > \frac{7}{3}$

    Case 2: $-(3x - 5) > 2$
    $-3x + 5 > 2$
    $3 > 3x$
    $x < 1$

14. What is the formal definition of continuity?
In calculus, continuity describes a function that has no breaks, jumps, or gaps in its graph at a given point or over an interval.
A function $f(x)$ is continuous at a point $x = c$ if three conditions are met:
    I. The function is defined at the point: $f(c)$ exists.
    II. The limit exists: The limit of $f(x)$ as x approaches c (from both sides) exists.
    III. The limit equals the function $\lim_{x \to c} f(x) = f(c)$.

15. Is $f(x) = (x - 2)^{\frac{2}{3}}$ continuous at $x = 2$?
Yes. The function is continuous even at corners and cusps.

16. Is the function $f(x) = \begin{cases} \frac{2x^2-7x+3}{x-3} & x \neq 3 \\ 5 & x = 3 \end{cases}$ continuous at $x = 3$?



Yes because the hole is filled in.

$f(x) = \frac{(2x-1)(x-3)}{x-3} \approx 2x - 1$

17. Given $f(x) = \sqrt{25 - x^2}$, find the domain in which $f(x)$ is continuous.

$-5 \leq x \leq 5$

18. Name the type of discontinuity:

a. $f(x) = \frac{x-2}{x^2-2x}$

$f(x) = \frac{(x-2)}{x(x-2)}$ (point discontinuity)

There also is a vertical asymptote at $x = 0$

Thus there is also an infinite discontinuity

b. $f(x) = \frac{|x-2|}{x-2}$

Jump discontinuity

c. $f(x) = \frac{1}{x+3}$

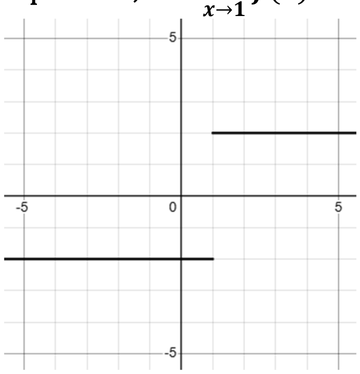
Infinite discontinuity

d. $f(x) = \sin\left(\frac{1}{x-1}\right)$

Oscillating discontinuity

19. $f(x) = \frac{2(x-1)}{|x-1|}$

a. If possible, find $\lim_{x \to 1} f(x)$



No limit

b. If possible, find $\lim_{x\to 1^-} f(x)$

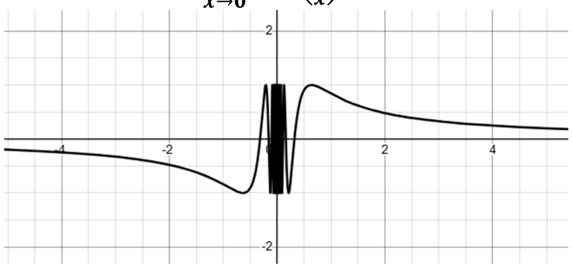
-2

c. If possible, find $\lim_{x\to 2} f(x)$

2

20. $f(x) = \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

a. If possible, find $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

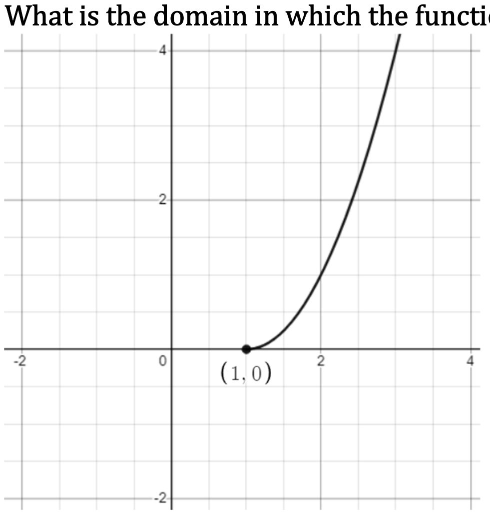


No limit

b. If possible, find $\lim_{x\to \infty} \sin\left(\frac{1}{x}\right)$

0

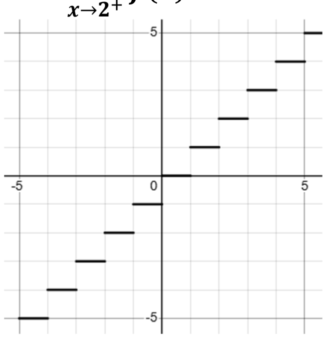
21. What is the domain in which the function below is continuous?



$x \geq 1$. At $x = 1$ we have endpoint continuity.

22. $f(x) = \lfloor x \rfloor$. This "Greatest Integer Function" is also known as floor(x) on Desmos.

a. Find $\lim_{x\to 2^+} f(x)$



2

b. Find $\lim_{x\to 2^-} f(x)$

1

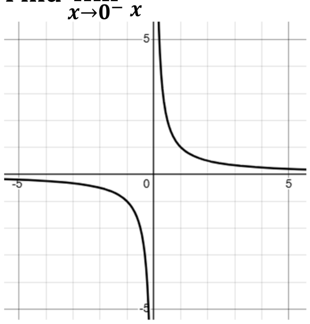
c. Find $\lim_{x\to -1^+} \lfloor x \rfloor$

$-\mathbf{1}$

d. Find $\lim_{x\to 2} f(x)$

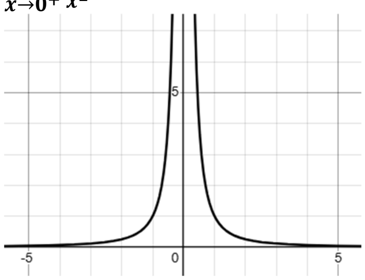
No limit

23. Find $\lim_{x\to 0^-} \frac{1}{x}$



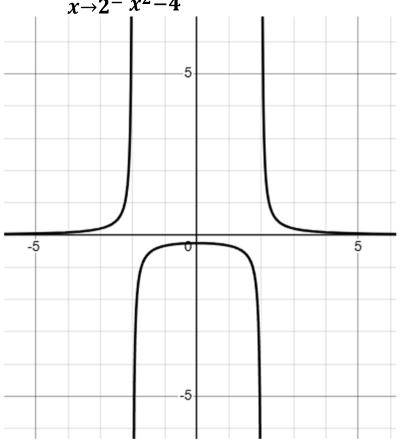
$-\infty$

24. $\lim_{x\to 0^+} \frac{1}{x^2}$



∞

25. Find $\lim_{x\to 2^-} \frac{1}{x^2-4}$

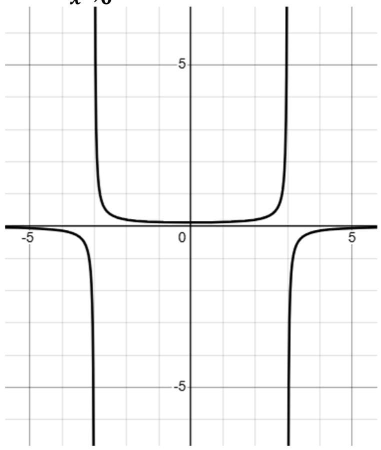


By knowing how to sketch the base function $f(x) = x^2 - 4$, we can roughly sketch the reciprocal function $\frac{1}{f(x)}$.

$-\infty$

26. $f(x) = \frac{1}{9-x^2}$

a. Find $\lim_{x\to 0} f(x)$



$\frac{1}{9-0^2} = \frac{1}{9}$

b. Find $\lim_{x\to\infty} f(x)$

0

c. Find $\lim_{x\to 3^-} f(x)$

$-\infty$

27. If $\lim_{x\to\pm\infty} f(x) = k$ then $y = k$ is a ______________ asymptote.

Horizontal

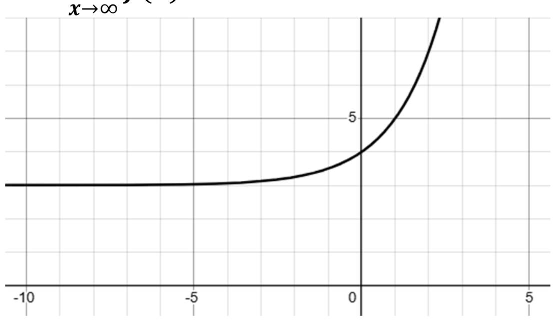
28. If possible, find $\lim_{x\to-\infty} \frac{6x-2}{2x+1}$

Divide coefficients of x:

$\lim_{x\to-\infty} \frac{6x-2}{2x+1} \approx \frac{6}{2} = 3$

29. $f(x) = 2^x + 3$.

a. Find $\lim_{x\to\infty} f(x)$

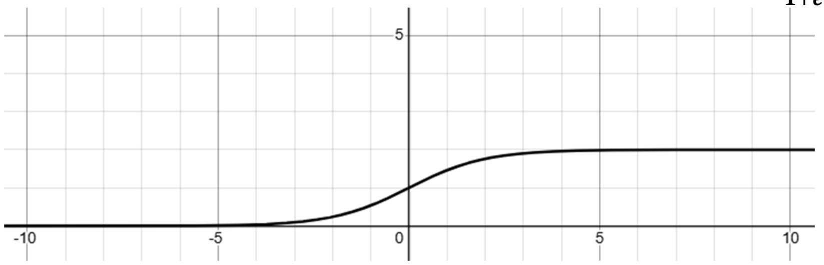


$= \infty$ (no limit)

b. $\lim_{x\to-\infty} f(x)$

3

30. Find the horizontal asymptotes of the logistic function $f(x) = \frac{2}{1+e^{-x}}$



$HA = \lim_{x\to\infty} \frac{2}{1+e^{-x}}$. As $x \to \infty$, $e^{-x} = \frac{2}{e^x}$ becomes very small approaching 0.

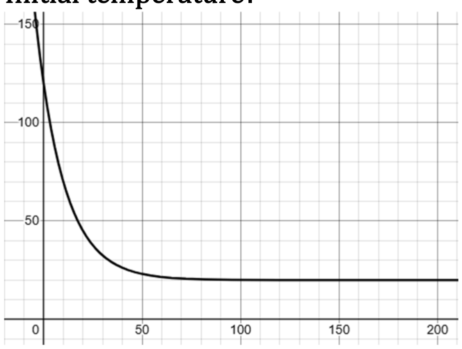
Thus $y = \frac{2}{1+0} = 2$.

$HA = \lim_{x\to-\infty} \frac{2}{1+e^{-x}}$. As $x \to -\infty$, $e^{-x} = e^{\infty}$ which becomes very large.

$\frac{2}{1+\infty} \to 0$. Thus $y = 0$.

31. Temperature $K(t) = 100\left(\frac{1}{2}\right)^{0.1t} + 20$

a. Initial temperature?



120

b. Describe the meaning of $\lim_{t\to\infty} K(t)$

Is the horizontal asymptote which is $K = 20$

32. Use the Squeeze / Sandwich Theorem to find:

a. $\lim_{x\to0} x^2 \sin\left(\frac{1}{x}\right)$

$-1 \le \sin A \le 1$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

Multiply each part by x^2

$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

Take the limit of each part:

$\lim_{x\to0}(-x^2) \le \lim_{x\to0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x\to0}(x^2)$

$0 \le \lim_{x\to0} x^2 \sin\left(\frac{1}{x}\right) \le 0$

$\lim_{x\to0} x^2 \sin\left(\frac{1}{x}\right) = 0$

b. $\lim_{x\to0}\left[x^3 \cos\left(\frac{\pi}{x}\right) + 2\right]$

Similar to the question above $x^3 \cos\left(\frac{\pi}{x}\right) = 0$

Thus $0 + 2 = 2$

33. If possible, find $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$

$$\lim_{x\to 0}\frac{1-\cos x}{x}\times\frac{(1+\cos x)}{x(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{\sin x}{x}\times\frac{x}{1+\cos x}=1\times 1\times\frac{0}{1+1}=0$$

34. If possible, find $\lim\limits_{x\to\infty}\frac{8x^6+3x^2+2x-1}{6x^2-5x+2}$

∞ (the highest power numerator outpowers the highest power denominator)

35. If possible, find $\lim\limits_{x\to\infty}\frac{9x^2-5x}{6x^5+3x^3-2x+1}$

0 (the highest power denominator outpowers the highest power numerator)

36. $\lim\limits_{x\to\infty}\frac{2x^3-5x^2+x}{4x^3+3x^2-7}$

$\approx\frac{2x^3}{4x^3}=\frac{1}{2}$ (the highest power numerator is tied with the highest power denominator)

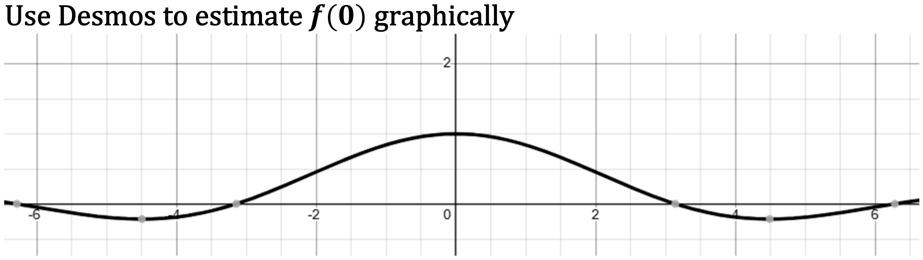
37. $f(\theta)=\frac{\sin\theta}{\theta}$

a. Use a table of values to estimate $f(0)$

x	$\frac{\sin x}{x}$
−2	0.45464871
−1	0.84147098
0	undefined
1	0.84147098
2	0.45464871
.1	0.99833417
.01	0.99998333
.001	0.99999983

1

b. Use Desmos to estimate $f(0)$ graphically



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38. $\lim\limits_{\theta\to\infty}\frac{\sin^2\theta}{\theta^2}$

$$\lim_{\theta\to\infty}\frac{\sin^2\theta}{\theta^2}=\lim_{\theta\to\infty}\frac{\sin\theta}{\theta}\times\frac{\sin\theta}{\theta}=0\times 0=1$$

39. $\lim\limits_{x\to 0}\frac{\sin 3x}{2x}$

$=\frac{3}{2}$ (coefficients of x)

40. $\lim\limits_{x\to\infty} \frac{\sin 3x}{2x}$

$\mathbf{0}$

41. $\lim\limits_{x\to 0} \frac{\sin 3x}{2}$

$= \frac{\sin(3\times 0)}{2} = \frac{0}{2} = 0$

42. $\lim\limits_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

$\lim\limits_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \times \left(\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}\right)$

$\lim\limits_{x\to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)} = \lim\limits_{x\to 7} \frac{(x-7)}{(x-7)(\sqrt{x+2}+3)} = \frac{1}{\sqrt{7+2}+3} = \frac{1}{6}$

43. $\lim\limits_{x\to 0} \frac{1-\cos x}{x}$

$\lim\limits_{x\to 0} \frac{(1-\cos x)}{x} \times \frac{1+\cos x}{1+\cos x} = \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$\lim\limits_{x\to 0} \frac{\sin x}{x} \times \frac{\sin x}{1+\cos x} = 1 \times \frac{0}{1+1} = 0$

44. $\lim\limits_{\theta\to\frac{\pi}{2}} \tan^2\theta\,(1-\sin\theta)$

Breaking $\tan\theta$ into $\frac{\sin\theta}{\cos\theta}$ and trig identities

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin^2\theta}{\cos^2\theta} \cdot (1-\sin\theta)$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin^2\theta}{1-\sin^2\theta} \cdot (1-\sin\theta)$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin^2\theta}{(1+\sin\theta)(1-\sin\theta)} \cdot (1-\sin\theta)$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin^2\theta}{(1+\sin\theta)} = \frac{(1)^2}{1+1} = \frac{1}{2}$

45. $\lim\limits_{\theta\to\frac{\pi}{4}} \frac{\cos 2\theta}{\sqrt{2}\cos\theta-1}$

$\lim\limits_{\theta\to\frac{\pi}{4}} \frac{2\cos^2\theta-1}{\sqrt{2}\cos\theta-1} = \lim\limits_{\theta\to\frac{\pi}{4}} \frac{(\sqrt{2}\cos\theta+1)(\sqrt{2}\cos\theta-1)}{\sqrt{2}\cos\theta-1} = \lim\limits_{\theta\to\frac{\pi}{4}}\left(\sqrt{2}\cos\theta+1\right) = \sqrt{2}\cos\left(\frac{\pi}{4}\right)+1$

$= \sqrt{2}\cdot\frac{\sqrt{2}}{2} + 1 = 2$

46. $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\sin^2(2\theta)}{1-\sin^2(\theta)}$

$\sin 2\theta$ identity Pythagorean identity

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{1-\sin^2(\theta)}$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{(2\sin\theta\cos\theta)^2}{1-\sin^2(\theta)}$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{4\sin^2\theta\cos^2\theta}{1-\sin^2(\theta)}$

$\lim\limits_{\theta\to\frac{\pi}{2}} \frac{4\sin^2\theta\,(1-\sin^2\theta)}{1-\sin^2(\theta)}$

$4\sin^2\left(\frac{\pi}{2}\right) = 4\times(1)^2 = 4$

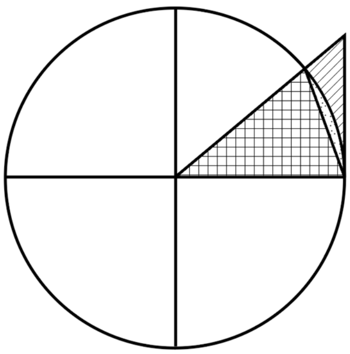
47. $\lim\limits_{x\to\infty} \frac{\sqrt{25x^2+25}}{x-6}$

Focus on $\sqrt{25x^2+25}$

$\sqrt{25(x^2+1)} = 5\sqrt{x^2+1} = 5 \cdot x \times \frac{1}{x}\sqrt{x^2+1} = 5x\sqrt{\frac{1}{x^2}(x^2+1)} = 5x\sqrt{1+\frac{1}{x^2}}$

$\lim\limits_{x\to\infty} \frac{\sqrt{25x^2+25}}{x-6} = \lim\limits_{x\to\infty} \frac{5x\sqrt{1+\frac{1}{x^2}}}{x-6} \approx \frac{5x\sqrt{1}}{x} \approx 5$

48. Enrichment: Use algebra and geometry to show that $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



Unit Circle

Khan academy proof: https://www.youtube.com/watch?v=5xitzTutKqM

Draw a line from the radius – the height of the triangle is $\sin\theta$

The height of the larger right triangle is $\tan\theta$

To make these triangle heights work in all quadrants use $|\sin\theta|$ and $|\tan\theta|$

In this justification we are only concerned with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Find area of small right triangle: $A = \frac{bh}{2} = \frac{1}{2}(1)\sin\theta$

(the base of the triangle is the radius of the unit circle $= 1$)

Find the area of the sector (wedge): $A_{sector} = \frac{\theta r^2}{2} = \frac{\theta(1)^2}{2} = \frac{\theta}{2}$. To make this work in all quadrants $A = \frac{|\theta|}{2}$

Now find area of the larger right triangle: $A + \frac{1}{2}bh = \frac{1}{2}(1)|\tan\theta| = \frac{|\tan\theta|}{2}$

Comparing the three areas:

$\frac{|\sin\theta|}{2} \le \frac{|\theta|}{2} \le \frac{|\tan\theta|}{2}$

Multiply by 2

$|\sin\theta| \le |\theta| \le \frac{|\sin\theta|}{|\cos\theta|}$

Divide each part by $|\sin\theta|$

$1 \le \frac{|\theta|}{|\sin\theta|} \le \frac{1}{|\cos\theta|}$

Now take the reciprocal of each part

$1 \ge \frac{|\sin\theta|}{|\theta|} \ge |\cos\theta|$

Absolute value signs are not necessary when θ is in Quadrant I. θ is positive and $\sin\theta$ is also positive in Quadrant I. When θ is negative in Quadrant IV, $\sin\theta$ is also negative. Two negatives make a positive. We also drop the absolute value sign from the $|\cos\theta|$ because x is positive in both QI and QIV.

Thus $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$

Now use the Squeeze / Sandwich Theorem

$$\lim_{\theta \to 0} 1 \le \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \le \lim_{\theta \to 0} \cos \theta$$
$$1 \le \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \le 1$$
$$\therefore \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$