Product and Quotient Rule Solutions

1. Product Rule Practice – Differentiate:
   a. $y = \sqrt[3]{x^2}(2x - x^2)$
   By expanding:
   $y = x^{\frac{2}{3}}(2x - x^2) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$
   $y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$

   Using the Product Rule: $f'g + g'f$
   Let $f(x) = x^{\frac{2}{3}}$ and $g(x) = 2x - x^2$
   $y' = \frac{2}{3}x^{-\frac{1}{3}}(2x - x^2) + (2 - 2x)x^{\frac{2}{3}}$
   $y' = \frac{2(2x - x^2)}{3x^{\frac{1}{3}}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{4x - 2x^2}{3x^{\frac{1}{3}}} +$
   $2x^{\frac{2}{3}} - 2x^{\frac{5}{3}}$
   $= \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$

   b. $P(x) = (6x^3 - x)(10 - 20x)$
   By expanding:
   $P(x) = 60x^3 - 120x^4 - 10x + 20x^2 =$
   $-120x^4 + 60x^3 + 20x^2 - 10x$
   $P'(x) = -480x^3 + 180x^2 + 40x - 10$

   Using the Product Rule: $f'g + g'f$
   Let $f(x) = 6x^3 - x$ and $g(x) = 10 -$
   $20x$
   $P'(x) = (18x^2 - 1)(10 - 20x) +$
   $(-20)(6x^3 - x)$
   $= 180x^2 - 360x^3 - 10 + 20x -$
   $120x^3 + 20x$
   $= -480x^3 + 180x^2 + 40x - 10$

2. $f(x) = e^x \cdot \sin x$. Find $f'(x)$
   $e^x \sin x + \cos x\, e^x = e^x(\sin x + \cos x)$

3. $y = \sin x\,(x^2)$
   $y' = \cos x\,(x^2) + 2x \sin x$

4. $y = e^x x^3$
   $y' = e^x x^3 + 3x^2 e^x$

5. $y = \ln x \cos x$
   $y' = \frac{1}{x}\cos x - \sin x \ln x$

6. $f(x) = \left(-5x^3 - 2x^{\frac{2}{3}} + 1\right)(x^2 + 5x)$
   Find $\frac{d}{dx}f(x)$
   $\left(-15x^2 - \frac{4}{3}x^{-\frac{1}{3}}\right)(x^2 + 5x) +$
   $(2x + 5)\left(-5x^3 - 2x^{\frac{2}{3}} + 1\right)$

7. $y = 2^x \ln x$. Find y'
   $y' = 2^x \ln 2 \cdot \ln x + \frac{1}{x} \cdot 2^x$

8. Find the derivative of $h(t) = (2\sqrt{t}) \cdot \sec t$
   $h(t) = \left(2t^{\frac{1}{2}}\right) \cdot \sec t$
   $h'(t) = t^{-\frac{1}{2}} \cdot \sec t + \sec t \tan t \cdot \left(2t^{\frac{1}{2}}\right)$

9. $y = 2x \arcsin x$
   Recall that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
   $y' = 2 \arcsin x + \frac{2x}{\sqrt{1-x^2}}$

10. $f(x) = (\tan^{-1}x) \cdot \tan x$
    Evaluate $f'\left(\frac{\pi}{4}\right)$
    $f'(x) = \frac{1}{1+x^2} \cdot \tan x + \sec^2 x \cdot \tan^{-1} x$
    $f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\frac{\pi}{4}\right)^2} \cdot \tan\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right) \cdot \tan^{-1}\left(\frac{\pi}{4}\right)$
    $\frac{1}{1+\frac{\pi^2}{16}} \cdot (1) + 2 \cdot \tan^{-1}\left(\frac{\pi}{4}\right) = \frac{16}{16+\pi^2} +$
    $2 \arctan\left(\frac{\pi}{4}\right)$

11. The Quotient Rule is derived in a similar manner to proving the Product Rule. State the Quotient Rule.
    $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

12. $W(z) = \frac{3z+9}{2-z}$
    Let $f(z) = 3z + 9$ and $g(z) = 2 - z$
    $W' = \frac{f'g - g'f}{g^2} = \frac{3(2-z) - (-1)(3z+9)}{(2-z)^2} = \frac{15}{(2-z)^2}$

13. $h(x) = \frac{4\sqrt{x}}{x^2-2}$
    Let $f(x) = 4x^{\frac{1}{2}}$ and $g(x) = x^2 - 2$
    $h'(x) = \frac{f'g - g'f}{g^2} = \frac{2x^{-\frac{1}{2}}(x^2-2) - 2x\left(4x^{\frac{1}{2}}\right)}{(x^2-2)^2} =$
    $\frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{(x^2-2)^2} = \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{(x^2-2)^2}$

14. $y = \frac{w^6}{5}$

It would make more sense to write this function as:

$y = \frac{1}{5}w^6$. Thus $y' = \frac{6}{5}w^5$

But we can also use the Quotient Rule: $\frac{f'g-g'f}{g^2}$

Let $f(w) = w^6$ and $g(w) = 5$

$\frac{dy}{dw} = \frac{6w^5(5)-(0)w^6}{5^2} = \frac{30w^5}{25} = \frac{6}{5}w^5$

15. $y = \frac{\log_2 x}{e^x}$

$\frac{f'g-g'f}{g^2}$

Remember $\frac{d}{dx}\ln x = \frac{1}{x}$

Also $\frac{d}{dx}\log_k x = \frac{1}{x \ln k}$

$y' = \frac{\frac{1}{x\ln 2}\times e^x - e^x \log_2 x}{e^{2x}} = \frac{1}{e^{2x}}\left[\frac{e^x}{x\ln 2} - e^x \log_2 x\right]$

$y' = \frac{1}{e^x e^x}\left[e^x\left(\frac{1}{x\ln 2} - \log_2 x\right)\right]$

$y' = \frac{1}{e^x}\left[\frac{1}{x\ln 2} - \log_2 x\right]$

16. $y = \frac{2^x}{\tan x}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}k^x = k^x \ln k$

$y' = \frac{f'g-g'f}{g^2}$

$y' = \frac{2^x \ln 2 \tan x - \sec^2 x 2^x}{\tan^2 x}$

17. $y = \frac{\sec x}{\tan^{-1} x}$

Recall that $\frac{d}{dx}\sec x = \sec x \tan x$ and

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$y' = \frac{\sec x \tan x \arctan x - \frac{\sec x}{1+x^2}}{(\arctan x)^2}$

18. $y = \frac{\ln x}{\pi \tan x}$

Recall that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}\tan x = \sec^2 x$

$y' = \frac{\frac{1}{x}\pi \tan x - \pi \sec^2 x \ln x}{\pi^2 \tan^2 x}$

Challenge

19. A classmate claims that $(f \cdot g)' = f' \cdot g'$ for any functions f and g.

Show an example that proves your classmate wrong.

Ex. $f(x) = x^2$ and $g(x) = x^3$

$(f \cdot g)' = (x^2 \cdot x^3)' = (x^5)' = 5x^4$

$f' \cdot g' = 2x \cdot 3x^2 = 6x^3$

$5x^4 \neq 6x^3$

20. Enrichment: Where does the Product Rule $(fg)' = f'g + g'f$ come from?

The product rule is also known as:

$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)]$

$= f'(x)g(x) + f(x)g'(x)$

We use the limit definition of derivatives.

Recall $m = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\frac{d}{dx}(f(x)g(x)) =$

$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$

Now split the quotient into two fractions:

$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$

$= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) +$

$\left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$

Can you recognize the limit definition of slope above?

$= f(x)g'(x) + g(x)f'(x)$